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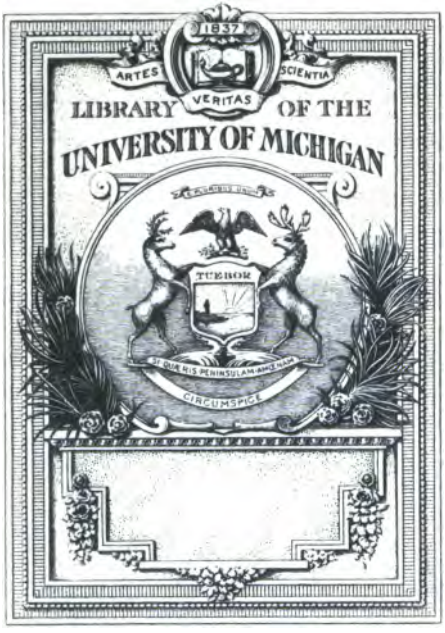

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Answer 24 (1)

Brief Course in Algebra

BY

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SYRACUSE, N. Y.

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ERRATA

P. 41 10th line for 3 read 2
19th line for 3 read 2
81 12th line for $-x^2$ read $+x^2$
133 last line for $-\frac{5}{8}$ read $=\frac{5}{8}$

TO

Mary Jane Manchester.

Math. Lib
12-31-1932

During the past year or two there has been a growing demand for a twenty weeks' course in Algebra which will not only serve as a preparatory course for more advanced mathematics, but will have an individual unity as well. It must offer an opportunity to the short term student to get a well established notion of the subject through quadratic equations.

This text is arranged to satisfy such a course. All unnecessary theorems, proofs, and processes have been omitted. The problem lists are very short and are to be supplemented by the instructor from the experience of the class. It is, in fact, a course in generalized arithmetic, arranged to face the demands made by the ninety per cent of students who expect to leave school either during or at the expiration of the high school course.

RAYMOND E. MANCHESTER

Oshkosh, Wis.
June 1, 1915

SECTION I

Treating of the relationship existing between Algebra and Arithmetic.

SECTION II

The Fundamental Operations.

SECTION III

Fractions.

SECTION IV

Powers and Roots.

SECTION V

Quadratic Equations.

Brief Course in Algebra

SECTION I

Treating of the relationship existing between Algebra and Arithmetic.

A. Letter Symbols
B. Negative Numbers
C. The use of the equation

Lesson 1

Algebra is an enlarged and continued course in arithmetic. As such it has a definite connection with the grade work in arithmetic and such elementary work in generalization as is attempted in the last year of grammar school.

The enlargement is (1) an addition to the Arabic symbol system by letters of the alphabet, (to stand for undetermined numbers); (2) an increase in the usefulness of the symbol system by using the plus (+) and minus (−)signs to qualify numbers (as well as for signs of operation); and (3) the use of the equation as a means of solving problems (made possible by the letter symbols). The study of arithmetical operations is continued with such revisement to the rules and laws as is necessary to cover the enlarged symbol system.

LETTER SYMBOLS

(1) Throughout the work in arithmetic, number ideas have been expressed by either a word (three),

a symbol (3); or a picture (111). But the word, the symbol, or the picture has stood for a definite number, so that the student, upon hearing the word, seeing the symbol, or seeing the picture, has had a distinct group brought to the attention. These names and symbols have been memorized with the connections always made clear.

(2) In any problem, however, there is a certain number (or there are in some cases certain numbers) not definitely known. There being no definite number idea, no name, Arabic symbol, or picture can stand for it. We know that the number, whatever it may be, exists, and we hope to determine its value by the solution of the problem. We speak of this number in various ways, such as, "the answer", "the unknown part", "the number to be found", or "the result".

Example—In a basket there are three times as many pears as apples. The total number of both is 28. To find the number of apples.

In this problem the unknown is the number of apples. We know that there is a certain number of apples, but cannot set down a symbol for the number until the problem has been solved.

Example—A certain number added to its half and its double gives a sum of 21. Find the number.

Here the unknown number is the number itself. We cannot put down a symbol to stand for it until the problem has been solved.

(3) In our course in algebra, we do have a symbol for such unknown numbers. It being impossible

to write down any one of the Arabic symbols until its value has been determined, it is necessary to use other symbols. Because of the familiarity of all students with the letters of the alphabet, it has become a custom to use one of these letters as a symbol until enough is known about the number to make it possible to use one of the Arabic symbols.

Example—

In the first problem mentioned above we might

Let x = the number of apples.

Then $3x$ = the number of pears.

Adding, $4x$ = the number of apples and pears.

Or, $4x = 28$.

Or, $x = 7$, the number of apples.

At the outset, we did not know that there were 7 apples, but we did know that there was a certain number of apples. We could not have used the symbol (3) or (5) or (9) or any of the other Arabic symbols with the certainty that the statement would be correct, so we allowed (x) to stand for the number, whatever it might be.

So it is that we may use the letters to stand for numbers until the numbers have been so established as to permit us to use the Arabic symbols.

(4) It follows that we may speak of (a) houses, (x) horses, (y) trees, or (b) books when there is no definite number idea in our minds. We know that there is a certain number of houses, or horses, or trees, or books, but we do not know what it is. There may be conditions given which will enable us to determine the number, and until such a determination is made the letters may be used.

The advantage of having such an addition to our symbol system is at once apparent when the solution of a difficult problem is attempted.

Example—A young man wishes to build a bookcase having 3 shelves arranged so that the lower one may contain larger books than the other two. Suppose he wishes to decrease each shelf space by two inches from the lower to the upper and still keep the bookcase in proportion. Given the inside measurement as 32 inches, to find the centers for the shelves.

Let x = space of the lower shelf.
then $x-2$ = space of the next shelf.
and $x-4$ = space of the upper shelf.
Adding, $x+(x-2)+(x-4)$ = total space.
Adding, $x+x-2+x-4=32$.
Or, $3x-6=32$.
Or, $3x=38$.
Or, $x=12\frac{2}{3}$ space of lower shelf.
Or, $x-2=10\frac{2}{3}$ space of middle shelf.
Or, $x-4=8\frac{2}{3}$ space of upper shelf.

(5) It is evident, therefore, that the letter stands for a general number while the Arabic symbol stands for the particular number. We may speak of (a) apples as well as (5) apples, providing we understand that the (a) is undetermined.

(6) If we are to use the letters to any great extent in number operations, it is necessary for us to review the arithmetical operations and readjust our rules and laws to accommodate the added symbols. Taking these operations in order, those of addition, subtraction, multiplication, and division

will be discussed first. The discussion will be very brief, however, owing to the fact that the operations are to be considered under separate topics.

Problems

1 What are the Arabic symbols?

2 What do they stand for?

3 Does the Arabic symbol stand for a particular or a general number?

4 When letters are used as symbols do they stand for particular or general numbers?

5 What does the expression ***generalized arithmetic*** mean?

6 Name other points of enlargement and generalization.

7 Can you state an advantage in having a general number symbol?

8 What are the definite number symbols?

9 Should the algebra be a continuation of the arithmetic?

10 Do you understand the distinction between a number and the symbol standing for it? Discuss this point.

11 Suppose $2x$ horses cost \$10. What would x horses cost?

12 If (a) boys are to have 12 apples divided among them, what is the share of each boy?

Lesson 2

Definitions and Rules

To facilitate discussion, the following definitions should be memorized.

1 *Multiplication* is expressed by using the dot (.), and, in the case of the presence of literal factors, by writing the factors together.

Ex. $3a \cdot 2 = 6a$ *Ex.* $a \times b = a \cdot b = ab$

Ex. $3a \cdot b = 3ab$ *Ex.* $x \cdot y \cdot z = xyz$

2 Any number or combination of numbers by signs of operation is called an algebraic *expression*.

Ex. $2a + b - 3$

a An expression of one term is called a *monomial*.

b An expression of two terms is called a *binomial*.

c An expression of three terms is called a *trinomial*.

d An expression of more than two terms is called a *polynomial*.

3 An algebraic expression not separated by the plus or minus sign is called a *term*.

Ex. $4ab$

$2a + b - 3$ is an expression made up of three terms.

4 Any one of a group of numbers is called a *factor* of their product.

Ex. $6ax$ has the factors 2, 3, a, x.

5 Any one of a group of equal factors is called a *root* of their product.

One of two equal factors is called the square root of their product.

One of three equal factors is called the cube root of their product.

One of four equal factors is called the fourth root of their product, etc.

6 A small number written at the upper right hand of a number is called an *exponent*, and expresses the fact that the number is taken a certain number of times as a factor.

Ex. a^2 The ^{2} expresses the fact that (a) is used twice as a factor.

7 Any factor or product of two or more factors is called the *coefficient* of the remaining factors.

Ex. $3a$ (3) is coefficient of (a)
(a) is coefficient of (3)

(*Note*) We usually refer only to the Arabic symbol as the coefficient.

Ex. $5x^2y$ 5 is coefficient of (x^2y)
$5x^2$ is coefficient of (y)
$5y$ is coefficient of (x^2)
x^2y is coefficient of (5)
x^2 is coefficient of $(5y)$
y is coefficient of $(5x^2)$

(*Note*) Ordinarily only (5) is referred to as the coefficient.

8 The product obtained by using a number two or more times as a factor is called a *power*.

Ex. $2 \cdot 2 \cdot 2 = 8$ (8) is the third power of (2)

Ex. $a \cdot a \cdot a \cdot a = a^4$ (a^4) is the fourth power of (a)

9 Signs of *aggregation* are those which are used to signify that certain expressions are to be considered as unified wholes. They are the parentheses (( )), the brackets ([ ]), the braces ($\{\ \}$), and the vinculum (—).

Example $2x+(3y+z)=2$

$3y+z$ is considered as a unified whole.

Example $6a+(2b-c)-3c=8$

$2b$—c is considered as a whole.

Example $(x-y)-4z$

$x-y$ is considered as a whole.

Example $(6m+n-p)+3d-4a$

$6m+n-p$ is thought of as a whole.

Example $(6a+b)-c=6$

$6a+b$ is here considered as a whole.

10 If the sign of aggregation is preceded by a (+) sign, the quantity within is added; if preceded by a (−) sign the quantity is subtracted.

11 In case the expression enclosed by the sign of aggregation is to be multiplied by a number, each term of the expression within is multiplied by the number.

Example $2(3x+4)=6x+8$

Example $a(x-y)=ax-ay$

Example $2x(c-d)=2xc-2xd$

12 The signs +, −, ×, ÷, −, are used as they have been used in arithmetic.

13 It often happens that letter symbols are given particular values in an expression to find the value of the expression.

Example—If $a=2$, $b=3$, $c=4$

To find the value of $3a+b-c+4ab+2b$

Substituting values,

$3(2)+(3)-(4)+4(2)\ (3)+2(3)$

Or, $6+3-4+24+6$

Or, 35

Problems

1 What are the signs of operation?

2 What are the signs of aggregation?

3 Define exponent, power, root.

4 Define coefficient, literal coefficient, numerical coefficient.

5 Define monomial, binomial, trinomial, polynomial.

6 What new symbol for multiplication has been introduced?

7 What is the sign of equality.

8 If a sign of aggregation includes certain terms of an expression, what does this signify?

9 How is a root indicated?

10 Do you find that words have the same meaning as in arithmetic?

11 Discuss fully the relationship existing between arithmetic and algebra.

Lesson 3

Discussion of the letter symbols with regard to the four fundamental operations.

Addition:

(1) If 4 is added to 3 it is understood that four units are to be grouped with three of the same kind whenever application is made to a problem. It is a matter of common agreement that the result is a group made up of 7 units. All Arabic symbols are names of certain groups.

(2) In abstract numbers the symbols are considered apart from particular objects, and the statement is made that $4+3=7$, a law which holds for all cases regardless of the objects considered.

(3) In adding letter symbols there is no possibility of saying that (a) units added to (b) units equals a group named by an Arabic symbol because of the fact that (a) and (b) stand for undetermined numbers.

(4) To add (a) and (b) then, it is only possible to indicate the sum $a+b$, but in case ($2a$) units are to be added to ($3a$) units, it is possible to write the sum as, $2a+3a=5a$. This last is true, owing to the fact that, even though (a) stands for an undetermined number, twice this undetermined number combined with three times this undetermined number equals five times the undetermined number. In the ex-

pressions $(2a)$ and $(3a)$, (a) is considered the common factor.

(5) We enlarge the rule for addition, therefore, to cover addition of numbers having letter factors, (literal factors). We say that numbers having like literal factors may be added by adding the coefficients of the literal factors and writing the sum as the coefficient of the common factor.

Example $5b+4b=9b$

(*Note*) (b) is common, so the coefficients (5) and (4) are added, giving the sum (9). This number is written as the coefficient of the common factor (b)

Example $6a+4a=10a$

Example $2ab+4ab=6ab$

(*Note*) Here the two factors (a) and (b) are common and may be considered as forming the product (a) times (b) or (ab), which product is common to both numbers. Therefore, twice the product added to four times the product equals six times the product.

Example $2a^2+4a^2=6a^2$

(*Note*) In this problem the exponent (2) indicates that the number (a) is taken twice as a factor, so that $2(a\cdot a)+4(a\cdot a)=6(a\cdot a)$ That is, two times $(a\cdot a)$ plus four times $(a\cdot a)$ equals six times $(a.a)$, or $6a^2$,—the 2 indicating that (a) is taken twice as a factor.

(6) The addition of expressions involving two or more terms is accomplished by arranging the terms so that the terms having like factors may be added together.

Example—Add $3a+4b+6c$
and $4a+2b+8c$
Sum is $7a+6b+14c$

(*Note*) Here it will be noticed that the numerical coefficients of the like factors are added and the sum written as the coefficient of the like factors.

Example Add $6xy+4w+8z^2$ and
$4w+3z^2+4xy$

This problem must be rearranged so that terms with like factors are in columns.

$6xy+4w+8z^2$
$4xy+4w+3z^2$
Adding, $10xy+8w+11z^2$

Problems

1 Add $6x$ $4b$ $8cb$
$4x$ $5b$ $+7cb$

Add $3a$ $6c$ $+4x$
$4a$ $10c$ $+8x$

2 Add $3xy$ $5cd$ $3mn$
$+4xy$ $4cd$ $+2mn$

Add $216a$ $649b$ $16abc$
$436a$ $36b$ $+4abc$

3 Add $+47x$ $16y$ $16y^2$
$+85x$ $+14y$ $11y^2$
$6y^2$

4 Add $1\frac{1}{2}y$ $6\frac{2}{3}x$
$2\frac{3}{4}y$ $4\frac{2}{5}x$

5 Add $6y$, $4x$, $8y$, $3x$, and $4y$
" mn, $4mn$, $6x$, $5y$, $3x$, and $2y$
" $3a$, $4b$, $6c$, $14a$, $10c$, and $18a$
" $4a$, $+6a$, $4b$, $13b$, and $+3b$
" $6d$, $+3d$, $4d$, $3c$, $4c$, and $+8c$

6 Add $3a+4b$ $2x+3y$ $4ab+7m$
$6a+3b$ $3x+2y$ $6ab+8m$

$15r+2s$ $18w+2x$ $c+10d$
$6r+3s$ $16w+3x$ $3c+6d$

7 Add $14l+3k+3m$
$16l+14k+6m$

Lesson 4

Discussion of the four fundamental operations. (Continued)

Subtraction:

(1) In arithmetic, to subtract (2) from (6), we determine the number which added to (2) will give (6)

(2) In algebra the operation is the same except that we can only subtract numbers having like literal factors. $(6a-3a=3a)$. We may define subtraction of numbers having literal factors by stating that numbers having like literal factors may be subtracted one from the other by subtracting the smaller coefficient from the larger and writing the difference as the coefficient of the common factor.

Example $(3x-2x=x)$

Example $(5xy-3xy=2xy)$

(*Note*) Here the common factor xy is the product of the factors x and y which are common to both terms.

Example From $5a+4b+6c$
Take $2a+3b+5c$
$3a+b+c$

(4) Arrangement must be made as in the case of addition so that the numbers having like factors may be in the same column.

Example From $6cd+4m^2+8p$
Take $4cd+3p+2m^2$

(*Note*) Here it is necessary to re-write so that the terms having like literal factors may be in the same column.

$$\begin{array}{ll} & 6cd+4m^2+8p \\ & 4cd+2m^2+3p \\ \text{Subtracting} & \overline{2cd+2m^2+5p} \end{array}$$

(5) It will be noted that the conditions for subtraction are the same as for addition.

(6) Terms having the same literal factors with the same exponents are called "like terms".

(7) Only like terms may be added or subtracted.

Problems

1 From 6 10 16 18 160
take 3 5 9 12 40

2 From $4a$ $16ab$ $6a^2b$ $9x^3y$ $5xy$
take $2a$ $3ab$ $4a^2b$ $10x^3y$ $7xy$

3 Subtract $4a+6cd+8d^2$ from $8a+10cd+4d^2$

4 Subtract $3x+4y^3+6zw$ from $6x+8y^3+10zw$

5 Subtract $2m+n+3rs$ from $12m+4n+8rs$

6 Subtract $16kl+23x^3y$ from $26kl+40x^3y$

7 Subtract $10a^2+3b^4+6cd+9e^3$ from $15a^2+8b^4+11cd+14e^2$

Lesson 5

Discussion of the four fundamental operations (continued)

Multiplication:

(1) To multiply numbers together having literal factors, it is only necessary to multiply the numerical coefficients together and indicate the multiplication of the literal factors by writing them together.

Example $2a \cdot 4b = 8ab$

Example $5cd \cdot 6cx = 30c^2dx$

(*Note*) The multiplication of these two numbers involves two (c) factors. Write 30 as the coefficient of the product of the literal factors $30cdcx$. There being two (c) factors, it is customary to write them as (c^2), showing that (c) is taken twice as a factor, or $30c^2dx$

Example $4ab \cdot 5a^3b^2 = 20a^4b^3$

(*Note*) Here $5 \cdot 4$ equals 20 for the numerical coefficient, and writing this number as the coefficient of the indicated product of the literal factors we have $20aba^3b^2$

Here it will be noted that (a) is taken a total of 4 times as a factor, and (b) a total of three times as a factor, so the product is written $20a^4b^3$

(*Note*) As stated before, the exponent indicates the number of times a number is used as a factor.

(2) It will also be noted that to multiply two like literal factors together it is necessary to add the exponents.

Example: $x^4 \cdot x^2 = x^6$

(3) In case an expression of two or more terms is to be multiplied by a single term, the process is as follows:

Example: Multiply $3x^2y+4xy^3+3w$
by $3xy$
$9x^3y^2+12x^2y^4+9xyw$

(*Note*) In this case each term of the larger expression is multiplied by $3xy$

Problems

Perform the following operations in multiplication.

1 Multiply $4a$ by 2
2 Multiply $16b$ by 3
3 Multiply $14c$ by $+12$
4 Multiply $+9ab$ by 7
5 Multiply $+11a^2c$ by $+8$
6 Multiply $4x+3y+z$ by $4x$
7 Multiply $3w+6z^2+4y$ by $5w+2z$
8 Multiply $16a^2+4b+c+2d$ by $7ab$
9 Multiply $+21m+3n^3+4p$ by $11+2m$

Lesson 6

Discussion of the four fundamental operations (continued)

Division:

1 Division is defined as the process of finding one of two factors when their product and one factor are given.

Example: Given the problem $8a^2b \div 2ab$

We may determine the quotient by establishing the number which when multiplied by $2ab$ will give $8a^2b$. This number is evidently the product of those factors of $8a^2b$ not contained in $2ab$.

$8a^2b = 2.2.2.a.a.b$

$2ab = 2.a.b$

$2,2.a$ = factors not contained in $2.a.b$

Therefore $4a$ is the quotient.

Example $10a^3b^2 \div 5ab$

Mentally it is possible to determine the quotient.

$10 \div 5 = 2$

$a^3 \div a = a^2$

$b^2 \div b = b$

therefore $10a^3b^2 \div 5ab = 2a^2b$

(*Note*) The fundamental operations are discussed fully in the next section.

2 To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

Example—$(6xy^2+4x^2y+8x^3y^4)\div 2xy$

$$2xy\,\big|\,\frac{6xy^2+4x^2y+8x^3y^4}{3y+2x+4x^2y^3}$$

Problems

1 Perform the indicated division.

$8x\div 2x$, $16y\div 8y$

$14a\div +7a$, $18xy\div +2xy$

$16a^2b\div 2a$, $24a^6b^4\div 8a^4b$

$18a^4b^6\div +9a^3b^4$; $40x^3y^3\div 10xy^2$

2 Perform the indicated divisions.

$$\frac{15xy}{3x}, \quad \frac{12a^2b}{6ab}, \quad \frac{12a^2bc}{+4ab}$$

$$\frac{+25c^4d}{5c^2d}, \quad \frac{+10m^4n}{+2m^3n}, \quad \frac{+180r^3s^2}{+90rs}$$

THE NEGATIVE NUMBER

Lesson 7

We have discussed the enlargement of the number system by the use of letter symbols. We will now discuss another enlargement of the number system not by the addition of new symbols but by the qualification of those we have.

It is possible to qualify number symbols so that the same symbols may have two meanings. This is accomplished by using the signs (+) and (−) *as signs of quality* as well as for signs of operation.

The sign (+) and (−) may be used *other than as signs of operation*. In such cases the sign is considered as a part of the symbol and is written with it. Let us take, for example, the reading of the thermometer. We are all accustomed to speak of degrees above or below zero, (zero being a starting point for measurement), so find it very convenient to indicate ten degrees above zero by writing the symbol (10) with the (+) sign preceding it, as follows (+10), and in like manner ten degrees below zero with the minus sign, (−10). Used in this way the signs do not indicate operations, but simply that measurement is made in one or the other of the two directions.

+10 degrees means ten degrees above zero.

−10 degrees means ten degrees below zero.

Another example would be in measurement of distance along a straight line with symbols for measure-

ment in one direction used with the (+) sign connected and symbols for measurement in the opposite direction with the (−) sign connected.

$$-3 \quad -2 \quad -1 \quad 0 \quad +1 \quad +2 \quad +3$$

The symbols stand for number groups as they do in arithmetic, but by the use of the (+) and (−) signs it is possible to see at a glance the particular group referred to. If (+2), we know that the measurement is in one certain direction we may agree upon, and if a (−2) the measurement is in the opposite direction.

Another example is to consider a man's assets as (+) and his liabilities as (−). $20 to be received would be written $+20, while ten dollars to be paid out would be written $−10. In this case it is evident that the man has ten dollars more receivable than he has payable. The balance would be $+10. The absolute value remains unchanged when these signs are used denoting quality. In order to avoid confusion, we will consider addition as combination.

In the following examples please note that this idea of combination is stressed.

Ex. Add $3b$ and $-4b$ Result $-b$

Ex. Add $5x$ and $-3x$ Result $+2x$

Ex. Add $15c$ and $2c$ Result $17c$

Ex. Add $-8d$ and $+d$ Result $-7d$

It follows that to express the combination of two or more numbers additively, it is only necessary to connect the numbers by the signs preceding them.

Ex. To combine $+8x$ and $-4x$

$$+8x-4x=4x$$

Here the combination of $8x$ and $-4x$ gives a balance in favor of the $(+)$ units of four units. To write

$$+8x-4x=4x$$

is equivalent to subtracting $4x$ from $8x$.

Ex. To combine $-5a$ and $2a$.

(*Note*) If no sign precedes the symbol it is understood that the sign is $(+)$.

$$-5a+2a=-3a$$

In such a problem as

$$3x-4x+7x-5x-8x$$

the process of combination calls for a combination of the $(+)$ units, a combination of the $(-)$ units, and finally a balancing of the two sums.

There are $(+10x)$ units and $(-17x)$ units, giving a balance of $(-7x)$ units.

Supposing $+5a-6b$ is to be added to $+4b-3a$

Arranging $5a-6b$

$$-3a+4b$$

Combining $2a-2b$

Here the balance between $(5a)$ and $(-3a)$ gives $(+2a)$, and the balance between $(-6b)$ and $(+4b)$ gives $(-2b)$.

It should be remembered that negative numbers, (minus numbers) are meaningful only in relation to positive numbers, (plus numbers). Just as (up) has no meaning except in relation to (down), or (east) except in relation to (west), or (right) except in relation to (left), so this negative number has no meaning

except in relation to the positive number. Inasmuch as they have these opposite meanings, to combine additively is merely balancing them.

(*Note*) It is assumed that only numbers having like literal factors may be so combined. All other combinations are simply indicated. *Ex.* To combine additively $(3a)$ and $(-4b)$ we simply write

$$3a-4b$$

Problems

1 Add $8a-2b+c-3d+5x$
$-6a+3b-6c+5d-8x$

2 Add $17cd-3ax+10cy$
$18cd+4ax-\ 8cy$

3 Add $7xyz+3ab-6cd$
$-4xyz-5ab+8cd$

4 Add $4a+\ 3b-\ 6c$
$-3a+\ 4b-\ 4c$
$-8a+14b-11c$
$+3a-\ 2b+\ 6c$

5 If a man's wealth is represented by $\$-500$ what is he worth after increasing it by $\$+800$?

6 If the temperature reading is $-5°$ what is it after a rise of $7°$.

7 A ship is at latitude $10°$. She sails south 640 miles. What is her latitude? (60 miles $= 1°$).

8 What is meant by
the temperature, $+3°$, $-6°$, $+18°$, $-20°$
the latitude, $+3°$, $-16°$, $+32°$, $-6°$, $10°$
the date, -162, $+1262$, 1896, -460

Lesson 8

Negative Numbers

Subtraction:

In arithmetical subtraction we simply diminish a group named by a symbol by some certain number of units.

Ex. $8-2=6$

Here the group of 8 units is diminished by 2 units, giving us a remainder of 6 units.

Suppose (-2) units is to be taken from $(+8)$ units. The meaning here is that the negative quality of the (2) units is taken from them ,thus restoring them to positive units. So to subtract (-2) units from $(+8)$ units the process is one of removal of the negative quality before combination takes place.

Thus $8-(-2)=10$

Or $7-(-4)=11$

Or $2a-(-3a)=5a$

Suppose it is desired to subtract $(+3)$ units from (-7) units.

$(-7)-(+3)=$

$-7-3=-10$

Ex. Subtract $2x-4y+c$ from $-3x+6y-8c$

Arranging terms so that those with like literal factors are in the same columns,

$$\begin{array}{r} -3x+6y-8c \\ +2x-4y+c \\ \hline -5x+10y-9c \end{array}$$

Combining, we have $(-3x)$ and $(-2x)$, giving $(-5x)$; $(+6y)$ and $(+4y)$, giving $(+10y)$; and $(-8c)$ and $(-c)$, giving $(-9c)$

Thus it is that the rule in algebra is to change the signs of the subtrahend and proceed as in addition.

Problems

1 From $5c$ $8ab$ $16ac$ $18bd$ $-15rs$
take $-4c$ $-4ab$ $-5ac$ $-8bd$ $-18rs$

2 From $18cd$ $-35ab$ $-43mn$ $25rs$ $-30xy$
take $-28cd$ $-40ab$ $60mn$ $-14rs$ $-43xy$

3 Two boys catch 490 fish. One boy catches fifty fewer than the other. How many does each catch?

4 How many years between the dates $+1462$ and -530?

5 How many years have elapsed since -622?

6 Discuss the absolute value of a number.

7 An elevator goes from the fourth floor to the basement. If distance above the main floor is $+$ and distance below is $-$, how would its distance from the main floor be represented?

8 What problems can you think of in which the signs $+$ and $-$ might be applied to the solution?

THE EQUATION

Lesson 9

(1) The equation is not a new thing. It has been used constantly in arithmetic, as for example,

$2+4=6$

$3 \cdot 2=6$, etc.

(2) Its usefulness has been greatly increased, however, by the use of the literal symbol. So efficient does the equation become as a method of solution that an eminent mathematician has said that the equation is the vital thing in algebra.

(3) The usefulness lies in the fact that by carrying through certain processes the unknown number may be expressed in terms of the known numbers, thus giving a solution.

Example: Suppose a problem is stated as follows: twice a number added to itself equals ten.

Let $x=$ number

$2x=$twice the number

Then $3x=10$

Or $x=10 \div 3$

Therefore $x=3\frac{1}{3}$

(4) The expressions upon the two sides of the equality sign are called the *members* of the equation.

(5) The members of an equation being equal, it follows that

(*a*) Equal numbers may be added to or subtracted from both members of an equation without changing the relationship.

(*b*) The members of an equation may be multiplied or divided by the same number without changing the relationship.

Example If $4=4$

Then $4+2=4+2$

Or $4-2=4-2$

Example If $4=4$

Then $4.2=4.2$ (multiplying by 2)

Or $4\div 2=4\div 2$ (Dividing by 2)

Problems

1 Why can the equation be used to greater advantage in Algebra than in Arithmetic?

2 Solve for x in the following equations.

$2x=8$

$4x=8+4$

$3x-2=8$

$4x+3=6+5$

$3+2x=9$

3 Solve for the letter symbol in the following equations.

$3a=9$

$4a+a=6+4$

$3a-2=8+6$

$2b-3b=4-b$

$3x+4=8$

4 A man had 180 apples in two baskets. In one basket he had twice as many apples as in the other. Find the number of apples in each basket.

(*Note*) Let the number of apples in one basket be x. In the other were $180-x$

5 Two boys had together 60 fish. One had 3 times as many as the other. How many had each?

6 The sum of two numbers is 105. One number is twice the other. Find the numbers.

7 A number added to six times itself equals 49. Find the number.

Lesson 10

The simple equation (one unknown number)

(6) If the unknown number appears in the equation with no higher exponent than (one), the equation is called a simple equation.

Example $2x+4=8$ ((x) has the exponent (1))

(7) The solution of a simple equation involving one unknown number consists of so applying the laws mentioned in Part **5** as to get an equation having the unknown number upon one side of the equality sign and known numbers upon the other side of the equality sign.

Example $6x+8=10$

Subtracting (8) from each number,

$6x+8-8=10-8$

Or $6x=2$

Dividing both members by (6)

$x=\frac{2}{6}$

Or, $x=\frac{1}{3}$

(8) The use of the equation makes possible the simple solution of many problems which otherwise might offer difficulty.

Example—

(9) Suppose I have three times as many German students as English students, and have a total of **140.** To determine the number of each,

Let $x=$number of English students

Then $3x$ = number of German students
Or $4x$ = total
Then $4x = 140$
And $x = 35$
And $3x = 105$

Ans. 35 English students.
105 German students.

(*Note*) This problem may be worked without using the symbol (x), but it is readily appreciated that to use the symbol (x) simplifies the solution.

(*b*) Three boys have a sack of apples to divide. The oldest boy is to have three times as many as the youngest, while the second oldest is to have twice as many as the youngest. How many does each receive if the sack holds 66 apples?

Let x = number of apples youngest boy receives
Then $2x$ = number of apples second boy receives,
And $3x$ = number of apples third boy receives.
Then $6x = 66$.
Or $x = 11$, number of apples youngest boy receives.
$2x = 22$, number of apples second boy receives.
$3x = 33$, number of apples third boy receives.

(*c*) Suppose four girls sell 620 Red Cross stamps. The second sells 60 more than the first, the third sells 10 more than the second, and the fourth sells 30 less than the first. To find how many each sells.

Let x = number first girl sells
Then $x+60$ = number second girl sells,
and $x+70$ = number third girl sells,
and $x-30$ = number fourth girl sells,
Or $4x+100$ = total number sold.

Then $4x+100=620$,

Or $4x=520$.

$x=130$, number first girl sells.

$x+60=190$, number second girl sells.

$x+70=200$, number third girl sells.

$x-30=100$, number fourth girl sells.

(*Note*) Any solution may be proved by substitution of the value of the unknown symbol in the original equation.

Example—Suppose $2x+4=8$

Then $2x=4$

and $x=2$

Substitution of (2) for (x) in the equation

$2(2)+4=8$

$4+4=8$

$8=8$

Problems

1 Solve the following equations for (x).

$3x+2a=4a$

$13x=26a$

$2b+2x=6b$

$2x-2c+3a=8c+6a$

$3a+b+x=2b$

2 Solve the following equations for (a)

$3a+2c=d$

$4a-3b=6b$

$6a+4c=b$

$6b+4x=-2a$

$y-4d+3a=2x$

3 If I pay 2 times as much for a hat as a shirt and five times as much for a suit of clothes as for the

hat and pay \$19.50 for all, how much do I pay for each?

4 If a man has a certain amount of money and adds (a) dollars and then has (c) dollars, how much did he have at first? (Solve in terms of (a) and (c).)

5 Two boys raked a yard. One received 40 cents less than the other. Together they received \$1.20. How much did each receive?

6 Two men go into business, The first puts in (a) dollars less than the other. Together they put in (d) dollars. How much has the first man put in the business? (Solve in terms of (a) and (d).)

7 A farmer raised twice as many bushels of oats as corn and three times as many bushels of potatoes as corn. In all he raised 1800 bushels. How many bushels of corn did he raise?

Lesson 11

The simple equation (two unknown numbers.)

(10) An equation containing two unknown numbers is called an *indeterminate* equation, owing to the fact than an indefinite number of pairs of values will satisfy the equation.

Example $2x-y=8$

Let $x=1$, then $2-y=8$, and $y=-6$.

Let $x=2$, then $4-y=8$, and $y=-4$.

Let $x=3$, then $6-y=8$, and $y=-2$.

(11) It follows that an equation containing two unknown numbers cannot be solved definitely.

(12) It is possible, however, to determine a pair of values providing a second condition is given upon which a second equation may be built.

Example Suppose	$2x+y=16$
Second condition	$x-y=2$
Adding equations	$3x=18$
Dividing by (3)	$x=6$

Substituting (6) in either equation (y) is found to be equal to (4)

(13) It follows that a solution may be determined for two equations involving the same two unknown numbers, but not for one equation considered singly.

(14) Such a solution is called the simultaneous solution of two indeterminate equations.

(15) *Examples:*

(*a*) To solve $\begin{cases} x+y=6 \\ 3x-y=4 \end{cases}$

Adding $4x=10$.

Dividing by 4, $x=2\frac{1}{2}$.

Substituting in the first equation, $2\frac{1}{2}+y=6$,

Or $y=3\frac{1}{2}$.

(*b*) $\begin{cases} 3a-2b=4 \\ 2a-2b=6 \end{cases}$

Subtracting the equations, $a=-2$

Substituting, $3(-2)-2b=4$, or $b=-5$

(*c*) $\begin{cases} 4c-2d=4 \\ 3c+3d=3 \end{cases}$

In this pair of equations it is noticed that neither addition nor subtraction of the equations will eliminate either unknown number. By multiplying the first equation through by (3) and the second through by (2), they become,

$$12c-6d=12$$
$$6c+6d=6$$

Adding $18c=18$

Dividing by (18), $c=1$

Substituting in the second equation, $d=0$

(*Note*) Solutions are proved by substitution of the values of the unknown symbols in the original equations.

Example—

Suppose $x+y=6$

$3x-y=2$

Adding, $4x=8$

Or $x=2$

Substituting. $2+y=6$, $y=4$

To prove, substitute these values in the original equations.

First equation, $2+4=6$, or $6=6$.

Second equation, $6-4=2$, or $2=2$.

Problems

1 In the equations used heretofore, how many unknown numbers have been involved? What is an indeterminate equation?

Determine five sets of values for each of the following equations.

2 $3x+2y=6$

3 $4x-3y=8$

4 $x+3y=16$

5 $-2x=10+4y$

6 $14=8x+2y$

7 If one set of values satisfies two equations, why is this set of values called the simultaneous solution of the two equations?

Solve the following pairs of equations simultaneously by the addition or subtraction method.

8 $3x+4y=8$
$2x+5y=3$

9 $3a+2b=6$
$4a-b=8$

10 $4y-2x=6$
$3y+x=4$

11 $2x=10-y$
$4x-y=8$

12 $2w-4z=6$
$4+2w=z$

13 $6x+4z=4$
$4z+3x=5$

Check results by substitution of the values in the original equations.

Lesson 12

Solve the following problems using the method explained in Lesson 11.

1 $4x-y=10$
$3x+y=4$

2 $5x+4y=8$
$x+y=2$

3 $3x+y=14$
$x+2y=6$

4 $3a+4b=6$
$4a=8+3b$

5 $3w+z=3$
$z=4+w$

6 $5a+4c=8$
$2c-4a=6$

Check results.

7 Find two numbers whose sum is 20 and whose difference is 4.

8 Two pounds of butter and three pounds of meat cost \$.50. Five pounds of butter and four pounds of meat cost \$2.30. Find the cost per lb, of butter and meat.

9 Twice the difference of two numbers is 4. Three times their sum is 42. What are the numbers?

10 In a certain section of Wisconsin the total value of the milk product exceeded the total value

of the wheat crop by $300,560. The two were valued at $422,081. Find the value of both.

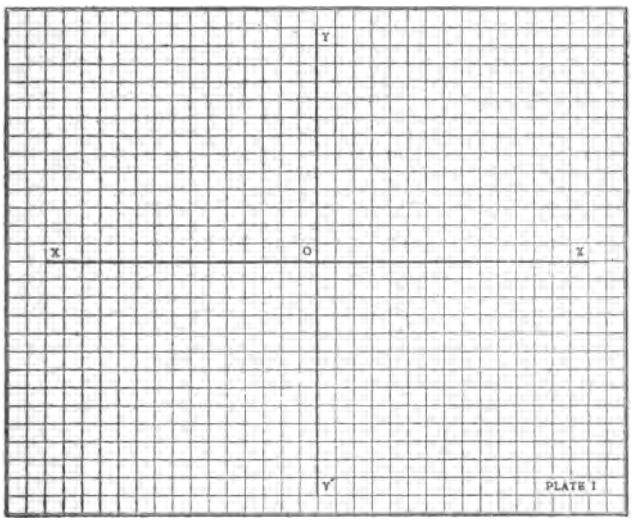
11 A house and lot are worth $4,230. The house is worth three times as much as the lot. What is each worth?

12 The sum of two numbers is 48. Their difference is 13. What are the numbers?

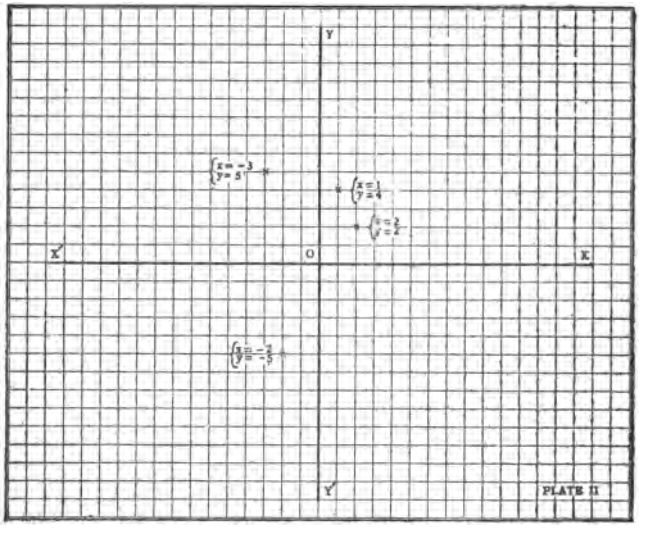
Lesson 13

Graphic representation

1 If two straight lines are drawn upon the plane perpendicular to each other they may be used as lines of reference for measurement. For instance, in the following figure we may consider that meas-



urement to the right of yy' is positive and measurement to the left of yy' is negative. Also, we may


Y
X'
O
X
Y'
PLATE II

consider measurement up from xx' is positive and measurement down as negative.

2 If the plane is divided in squares and x values measured to the right or left with y values measured up or down, points may be located having given an x and y value for each.

Ex. To locate the points represented by

(1) $x=3$ $y=2$ (2) $x=1$ $y=4$ (3) $x=-3$ $y=5$ (4) $x=-2$ $y=-5$

(1) To locate $\begin{pmatrix} x=3 \\ y=2 \end{pmatrix}$ measure three units to the right of the intersection on the xx' line, then two units up. This locates the point. (2) To locate the point $\begin{pmatrix} x=1 \\ y=4 \end{pmatrix}$ measure one square to the right, then four squares up. (3) To locate the point $\begin{pmatrix} x=-3 \\ y=5 \end{pmatrix}$ measure three squares to the left, then five squares up. (4) To locate the point $\begin{pmatrix} x=-2 \\ y=-5 \end{pmatrix}$ measure two squares to the left, then five squares down.

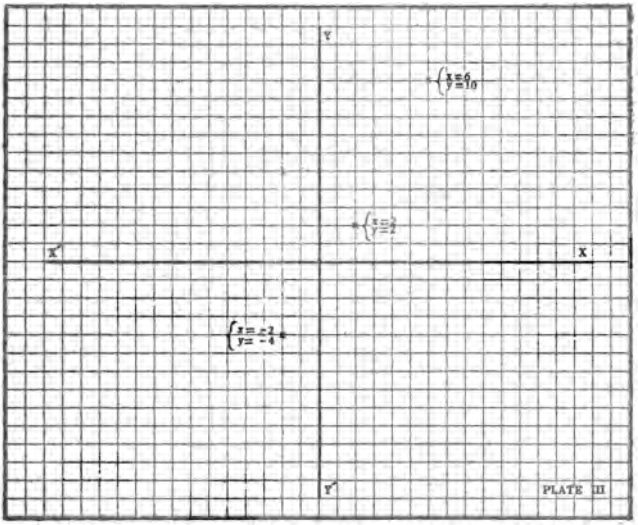
Ex. Locate the following points.

(1) $x=3$ $y=2$ (2) $x=6$ $y=10$ (3) $x=-2$ $y=-4$

Problems

Locate the following points after drawing the lines xx' and yy'.

$x=6$	$x=-2$	$x=-3$	$x=3$
$y=2$	$y=5$	$y=-4$	$y=-6$


Y
X'
X
Y'
PLATE III

Lesson 14

Representation of equations

From any given indeterminate equation any number of pairs of values may be found satisfying the equation.

Ex. Given the equation $x+2y=6$

If $x=1$, $y=\frac{5}{2}$	If $x=-1$, $y=\frac{7}{2}$
If $x=2$, $y=2$	If $x=-2$, $y=4$
If $x=3$, $y=\frac{3}{2}$	If $x=-3$, $y=\frac{9}{2}$
If $x=4$, $y=1$. etc.	If $x=-4$, $y=5$. etc.

If these values of x and y are considered as representing points, then it is possible to get a graphical representation of the equation.

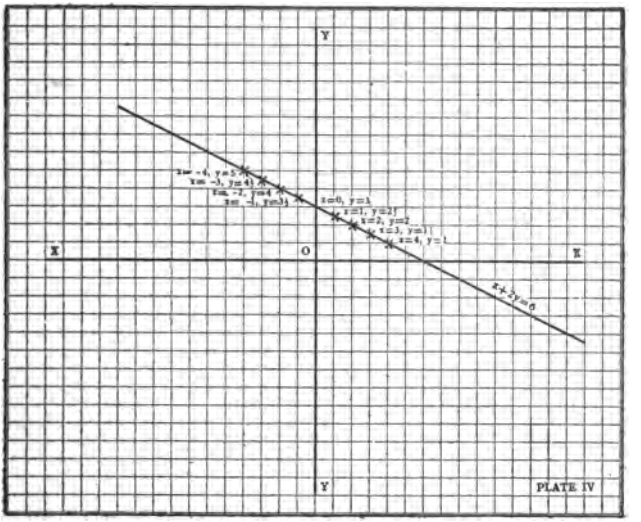
A line drawn through these points represents the equation.

In the above problem only a few of the infinite pairs of values were located, but enough were located to lead one to conclude that the line joining all the points would be straight.

Problems

By the above method plot the following equations.

1 $x+y=2$	3 $2x+3y=5$
2 $x-y=8$	4 $x-5y=6$


Y
X
O
X
Y
x+y=6
PLATE IV

Lesson 15

Representation of two equations

The lines representing two equations will intersect at some point except when the lines are parallel. This point of intersection is the only point that is common to the two lines; therefore, it represents the simultaneous solution of the two equations.

Ex. Given $3x+y=9$
$4x-y=5$

To solve simultaneously by graphical representation.

First, plot the lines representing the equations.

Second, determine the exact point of their intersection. This is the graphical solution.

In the above problem it is noticed that the lines intersect very near a point represented by the values $x=2$, $y=3$. If the equations are solved simultaneously, the values of x and y are 2 and 3. It is evident that there is correspondence between the graphical and the simultaneous solutions.

Problems

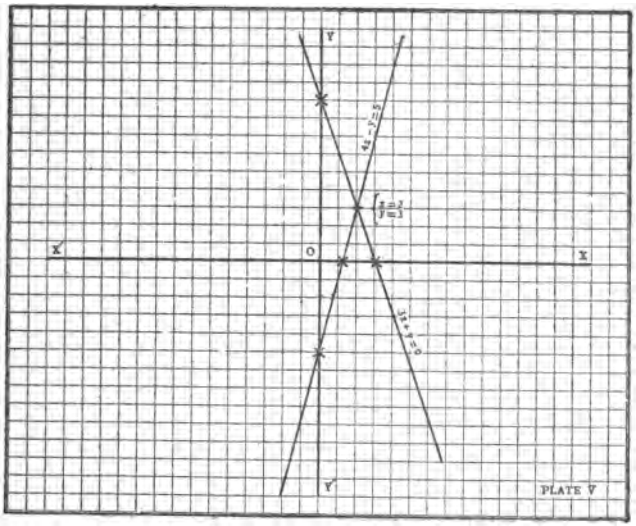
Solve graphically the following pairs of equations. Then solve algebraically and compare results.

1 $x+y=4$
$x-y=2$

2 $2x+y=3$
$4x-y=3$

3 $5x+y=16$
$x-y=2$

4 $x+2y=5$
$3x+4y=11$


Y
X
O
X
Y'
$\begin{cases} x=2 \\ y=3 \end{cases}$
PLATE V

Lesson 16

A very short method for plotting an equation is to give to each unknown in turn the value zero.

Ex. $2x+y=3$

First, let $x=0$, then $y=3$

Second, let $y=0$, then $x=\frac{3}{2}$

These values represent points on the axes.

If $x=0$, the y measurement is upon the y axis.

If $y=0$, the x measurement is upon the x axis.

Inasmuch as two points determine a line, the graph is determined.

Problems

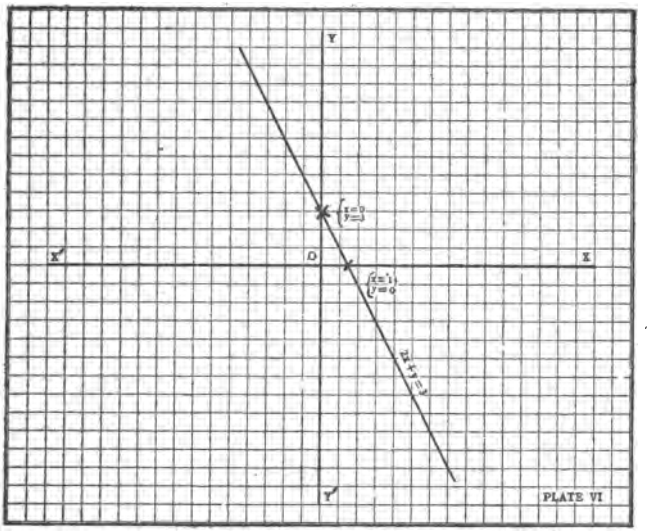
Plot the following equations by determining where the equations intercept the axes.

1 $x+y=6$

2 $2x+y-5=0$

3 $x-2y=8$

4 $3x-2y=9$


Y
X'
O
X
Y'
2x + y = 2
PLATE VI

Lesson 17

Plotting statistics

1 Suppose the school attendance over a term of years is represented by the following table.

1905—496
1906—500
1907—480
1908—510
1909—525
1910—550
1911—580
1912—590
1913—615

The attendance curve may be constructed as follows.

650
640
630
620
610
600
590
580
570
560
550
540
530
520
510
500
490
PLATE VII

29
18
15
10
Fig. I
Fig. II
5 hours
4 hours
3 hours
2 hours
1 hour
Chicago
Meeting point
Rockford
PLATE VIII

2 It is desired to get a mixture of milk and cream which shall contain 18% fat, from cream testing 28% and milk testing 3%

In the upper left hand corner of the rectangle place the number representing precentage of fat in the cream. In the lower left hand corner place the number representing the percentage of fat in the milk. In the centre of the rectangle place the number representing the percentage to be found. By cross subtraction it is found that 15 parts of cream are to be taken and 10 parts of milk are to be taken.

3 A train leaves Chicago toward Rockford running 40 miles per hour. A train leaves Rockford at the same time toward Chicago running 30 miles per hour. At the end of one hour, which would mean measurement of one unit up, the train from Chicago has gone 40 miles, represented by measurement of four units to the right. Thus a point is located which forms a straight line with the original point. Measuring from the opposite direction one unit up and three to the left, a point is located which forms a straight line with the original point. The intersection of these lines represents the meeting point of the two trains. The distance up represents time. The distance to the right and left represents distance from the two cities.

Note Chicago and Rockford are 90 miles apart.

SECTION II

The Fundamental Operations

ADDITION

Lesson I

(1) The addition of polynomials depends directly upon the addition of monomials. All that is necessary is to arrange terms so that similar terms are in columns.

Ex. To add $3a+2b$, $6a+4b$

Arranging, $3a+2b$
$6a+4b$

Adding, $9a+6b$

Ex. To add $4xy-6w$, $16w-3xy$

Arranging, $4xy-6w$
$-3xy+16w$

Adding, $xy+10w$

Ex. To add $3ab-2c+4d$, $3ab+6c-2d$ and $4c+3d-2ab$

Arranging, $3ab-2c+4d$
$3ab+6c-2d$
$-2ab+4c+3d$

Adding, $4ab+8c+5d$

Ex. To add $2x-3y+4z$, $-3x+2y-2z$, and $5x+2y-3z$

Arranging, $2x-3y+4z$
$-3x+2y-2z$
$5x+2y-3z$

Adding, $4x+y-z$

Problems

1 Add $6x+4y$
$-3x-2y$

2 Add $3w+4z$
$6w-2z$

3 Add $2a+4b-c$
$-3a+2b-2c$

4 Add $2ab+5cd$
$3ab+8cd$

5 Add $5xyz-2ab$
$6xyz+3ab$
$2xyz-5ab$

6 Add $5x^2y+6w$
$-3x^2y-2w$

7 Add $15m-5n$
$2m+3n$
$12m-2n$

Lesson 2

(2) Polynomials having literal or mixed coefficients may be added by indicating the sum of the coefficients for a new coefficient.

Ex. To add, $ax+by$, $cx-dy$, and $4x+3y$

Arranging, $ax+by$
$cx-dy$
$4x+3y$

Adding, $(a+c+4)x+(b-d+3)y$

(*Note*) In this problem the letters x and y stand for the variables or unknown numbers.

(3) Often terms are to be added involving binomial variables.

Ex. To add, $3(a+x)+6(a+x)-2(a+x)+5(a+x)$

Arranging, $3(a+x)$
$6(a+x)$
$-2(a+x)$
$5(a+x)$

Adding, $12(a+x)$

Ex. To add, $2x+(a+b)$, $4x-2(a+b)$, and $-3x-6(a+b)$

Arranging, $2x+(a+b)$
$4x-2(a+b)$
$-3x-6(a+b)$

Adding, $3x-7(a+b)$

Ex. To add $6m+3n-(x-2y)$, $3m-2n+6(x-2y)$, and $2m+n$

Arranging, $6m+3n-(x-2y)$
$3m-2n+6(x-2y)$
$2m+n$

Adding, $11m+2n+5(x-2y)$

(4) In case expressions have terms connected and held together by signs of aggregation, a removal of the signs of aggregation does not affect the signs of the terms, provided the signs of aggregation are preceded by the plus (+) sign.

Ex. $3x+(4y-3c+d)$

Removing the parentheses,

$3x+4y-3c+d$

Ex. $2a-6b+(4x+y-3a)$

Removing the parentheses,

$2a-6b+4x+y-3a$

Problems

1 Add $ay+bz$, $2cy-6z$

2 Add $5x+aw$, $bx+cw$, $-2x-5w$

3 Add $6a+4(x-y)-6(x+y)$, $10a-5(x-y)-2(x+y)$, $2a-10(x-y)+3(x+y)$

4 Combine $6x-4(a+b)+3c+5x+6(a+b)-4c+8(a+b)$, $-2x$

5 Combine $15(x+y)+20c-10(a-b)+2(x+y)-6c+4(x+y)-2c$

SUBTRACTION

Lesson 3

(1) Algebraic subtraction is the process of determining what must be added to the subtrahend to give the minuend.

Ex. From $10x$ take $3x$

To find what must be added to $3x$ to give $10x$

It is evident from arithmetical subtraction that the answer is $7x$.

Ex. From $14a$ take $2a$

Answer, $12a$

Ex. From $6mn$ take $3mn$

Answer, $3mn$

(2) By reference to the discussion of the negative number in Section I, it is evident that the subtraction of a larger number from a smaller gives a negative number for a difference.

Ex. From 7 take 9

The answer then is -2

Ex. From $2a$ take $6a$

If $-4a$ is added to $6a$ the result is $2a$. The answer then is $-4a$.

Ex. From $3b$ take $12b$

If $-9b$ be added to $12b$ the result is $3b$. $-9b$ is the answer.

(3) The working rule for subtraction, when both positive and negative numbers are involved, is to change the sign of the subtrahend and proceed as in addition.

Ex. From $14cd$ take $3cd$.

Changing the sign of the term $3cd$, it becomes $-3cd$.

Adding $14cd$ and $-3cd$, we get $11cd$

Therefore $14cd-3cd=11cd$

Ex. From $3xy$ take $2xy$

Changing the sign of the subtrahend, $2xy$ becomes $-2xy$

Adding $3xy$ and $-2xy$, we get xy

Therefore $3xy-2xy=xy$

Ex, From $6bc$ take $-2bc$

Changing the sign of the subtrahend, $-2bc$ becomes $+2bc$.

Adding $6bc$ and $+2bc$, we get $8bc$

Therefore $6bc-(-2bc)=8bc$

(*Note*) From the first general rule it is evident that it is necessary to add $+8bc$ to $-2bc$ to give $6bc$

Ex. From $-8xy$ take $-5xy$

Changing the sign of the subtrahend, $-5xy$ becomes $+5xy$

Adding $-8xy$ and $+5xy$, we get $-3xy$

Therefore, $-8xy-(-5xy)=-3xy$

(*Note*) From the general rule for subtraction it is evident that it is necessary to add $-3xy$ to $-5xy$ to get $-8xy$

Ex. From $2(c+d)$ take $-3(c+d)$

Changing the sign of the subtrahend, $-3(c+d)$ becomes $+3(c+d)$

Adding $2(c+d)$ and $+3(c+d)$ the result is $5(c+d)$

Therefore, $2(c+d)-(-3(c+d))=5(c+d)$

Ex. From $-8pq$ take $6pq$

Changing the sign of the subtrahend, $6pq$ becomes $-6pq$

Adding $-8pq$ and $-6pq$ we get $-14pq$

Ex. To simplify $3x+4y-2x+y-6x$

Collecting terms,

$$\begin{array}{ll} +3x & +4y \\ -2x & +\ y \\ -6x & \\ \hline -5x & +5y \end{array} \qquad \text{Or } -5x+5y$$

(*Note*) In such a problem the process is one of combination of terms with consideration for the signs preceding them. Although the negative sign is involved, the problem is simplified by direct reference to the method of addition. (Addition is defined as combination of terms.) It follows that the presence of the negative sign does not always mean that the process of algebraic subtraction is indicated. In all cases where mere combination is to be made, the process is one of algebraic addition, even though the negative sign is involved.

Problems

1 Subtract $6a$ from $12a$

2 Subtract $2x$ from $6x$

3 Subtract $7c$ from $5c$

4 Subtract $15xy$ from $8xy$

5 Subtract $4mn$ from $5mn$

6 Subtract $6a$ from $-10a$

7 Subtract $-5xy$ from $-8xy$

8 Subtract $6(a+b)$ from $10(a+b)$

9 Subtract $12(a-c)$ from $7(a-c)$

10 Subtract $-15c$ from $2c$

11 Combine $3a+5b-2c+8a-4c-3b-18a+2b-c$

12 Combine $2x+y-6x+4y-3x+4y-7x+8y-3y$

Lesson 4

Subtraction of polynomials

(4) The process of subtraction of one polynomial from another is performed by application of the preceding rules to each group of similar terms.

Ex. From $3x+2y$ take $2x-3y$

Arranging, $3x+2y$

$2x-3y$

Subtraction of similar terms is accomplished by changing the sign of each sign of the subtrahend, and proceeding as in addition.

Changing terms of the subtrahend,

$3x+2y$

$-2x+3y$

Adding, $x+5y$

The process is one of (1) arranging terms so that similar terms are in columns, and (2) applying the rule for subtraction to each group of similar factors.

The terms of an expression may be rearranged so that similar terms are in columns without altering the value of the expression.

Ex. From $3x+2y$ take $3y+6x$

Arranging, $3x+2y$

$6x+3y$

Subtracting, $-3x-y$

Ex. Perform the following indicated operation.

From, $2a+3b$

Take, $a-2b$

Changing the signs of the subtrahend,

$2a+3b$

$-a+2b$

Adding, $a+5a$

Ex. Perform the following indicated operation.

From, $6x+2y-3z$

Take, $-4x+3y+4z$

Changing the signs of the subtrahend,

$6x+2y-3z$

$+4x-3y-4z$

Adding, $10x-y-7z$

Ex. Perform the indicated operation.

From, $8xy-4w+3m^2n$

Take, $2xy-3w-2m^2n$

Changing the signs of the subtrahend.

$8xy-4w+3m^2n$

$-2xy+3w+2m^2n$

Adding, $6xy-w+5m^2n$

Problems

1 Take $6b-4cd+8m$ from $3b-5cd-10m$

2 Take $18cd^2+19m^3n$ from $14cd^2-11m^3n$

3 Take $6a+3b-3c$ from $8a-4b+7c$

4 Take $5x-4y+6z$ from $x-2y-8z$

5 Take $16vw-8y+6z^2$ from $8vw-10y-18z^2$

6 From $18a-14d^2+c$ take $-10a+8d^2-3c$

7 From $42xy+3z+14d$ take $14xy-13z-18d$

8 From $25m^5n+4rs^2$ take $16m^5n-8rs^2$

9 From $4ab-6cd+14ef$ take $7ab+9cd-5ef$

10 From $2a-3cd^4$ take $-5a+7cd^4$

Lesson 5

(5) Expressions involving literal coefficients for the unknown numbers may be subtracted by indicating results.

Ex. Suppose x and y are the unknown numbers.

From, $ax+by$

Take, $cx-dy$

Changing the signs of the subtrahend,

$$ax+by$$
$$-cx+dy$$

Adding, $(a-c)x+(b+d)y$

Ex. Suppose m and n are the unknown numbers.

From, $rm+sn$

Take, $-pm-qn$

Changing the signs of the subtrahend,

$$rm+sn$$
$$pm+qn$$

Adding $(r+p)m+(s+q)n$

(6) Often the coefficients of the unknown numbers are mixed numbers. In such cases the process of subtraction follows directly the process of addition after the signs of the subtrahend are changed.

Ex. Suppose x and y are the variables.

From, $3ax-2by$

Take, $-7ax+3by$

Changing the signs of the subtrahend,

$$\begin{array}{r} 3ax-2by \\ +7ax-3by \\ \hline \end{array}$$

Adding, $10ax-5by$

Ex. Suppose c and d are the unknown numbers.

From, $3ac+4bd$

Take, $2xc+3yd$

Changing the signs of the subtrahend,

$$\begin{array}{r} 3ac+4bd \\ -2xc-3yd \\ \hline \end{array}$$

Adding, $(3a-2x)c+(4b-3y)d$

(7) Binomial and polynomial terms are treated as monomial terms are treated, the coefficients alone being considered.

Ex. From, $2(x-y)+4(a+b)$

Take, $-3(x-y)+3(a+b)$

Changing the signs of the subtrahend,

$$\begin{array}{r} 2(x-y)+4(a+b) \\ 3(x-y)-3(a+b) \\ \hline \end{array}$$

Adding, $5(x-y)+(a+b)$

Ex. From $3a(m+n)-6b(r-s)$

Take $-2a(m+n)-3b(r-s)$

Changing the signs of the subtrahend,

$$\begin{array}{r} 3a(m+n)-6b(r-s) \\ 2a(m+n)+3b(r-s) \\ \hline \end{array}$$

Adding, $5a(m+n)-3b(r-s)$

Ex. From, $6ab(p+q+r)+3a(m-y)$
Take, $-2cd(p+q+r)+ w(m-y)$
Changing the signs of the subtrahend,
$6ab(p+q+r)+3a(m-y)$
$2cd(p+q+r)- w(m-y)$
Adding, $(6ab+2cd)\ (p+q+r)+(3a-w)\ (m-y)$

Problems

1 From $12a$ take $5a$
2 From $2ax$ take ax
3 From ax take bx. (x is the variable)
4 From my take ny (y is the variable)
5 From abx take cdx (x is the variable)
6 Subtract $5(a+b)$ from $10(a+b)$
7 Subtract $6(x-y)$ from $4(x-y)$
8 Subtract $-18(c-d)$ from $20(c-d)$
9 From $6\sqrt{ab}$ take $4\sqrt{ab}$
10 From $8\sqrt{a}$ take $6\sqrt{a}$
11 From $18(a+b-c)$ take $9(a+b-c)$
12 From $6(x-2y+d)$ take $8(x-2y+d)$
13 From $3(x+y)$, $6\sqrt{ab}$, $18(x+y-z)$
Take $-6(x+y)$, $-7\sqrt{ab}$, $-24(x+y-z)$
14 From $3x^2y^2z^3$, $-15\sqrt{abc}$, $+18\ (m-n)$
Take $-7x^2y^2z^3$, $-28\sqrt{abc}$, $-25(m-n)$

Lesson 6

Removing signs of aggregation

(8) If signs of aggregation, preceded by the negative sign, are to be removed, all the signs within the signs of aggregation are changed.

Ex. $3ab-2c-(2ab+3c)$

To remove the parentheses, all signs of terms enclosed must be changed.

$3ab-2c-2ab-3c$

Collecting terms, $3ab-2ab-2c-3c$

Or, $ab-5c$

(*Note*) It is evident that the presence of the negative sign preceding the sign of aggregation signifies that the terms within are to be subtracted. From the working rule the signs of the subtrahend are changed, and we proceed as in addition.

Ex. $3ab-2c-(6x-2ab+3c)$

Removing the parentheses, and changing all signs within,

$3ab-2c-6x+2ab-3c$

Collecting terms, $3ab+2ab-2c-3c-6x$

Or, $5ab-5c-6x$

(9) Often there are signs of aggregation within signs of aggregation. To remove them the same rule applies.

Ex. $3a+[3x+4b-(6a-2x+b)-2a]$

Removing the parentheses,

$3a+[3x+4b-6a+2x-b-2a]$

Removing the brackets,

$3a+3x+4b-6a+2x-b-2a$

Collecting terms,

$3a-6a-2a+3x+2x+4b-b$

Or, $-5a+5x+3b$

(*Note*) It is customary to remove the innermost sign of aggregation first.

Problems

1 Simplify $16x-(4a-b)$

2 Simplify $8+6a-(4b-c)$

3 Simplify $15ef-(6a+3b)-(4b-d)$

4 Simplify $13p-(6r+4)$

5 Simplify $15a+13b-4d-d+c$

6 Simplify $16a-(a-b)+[c-d]-\{a+d\}$

7 Simplify $a-[b+(c+d)]$

8 Simplify $x+y-(a+b-(m-n))$

9 Simplify $4x+\{-4a+y-(6c+d-x)+3y-(4a+c)\}$

Enclose the following expressions with signs of aggregation preceded by the negative sign.

10 $6a+4b-c$

11 $8m+16n-y-3r$

Enclose the last three terms in parentheses preceded by the negative sign.

12 $4d-3c+5x-3y-4n$

13 $18r^2s+5mn-16a-4c$

14 $2a-4b-16c+18d+m$

MULTIPLICATION

Lesson 7

General definitions.

(1) Multiplication is the process of performing upon a number (called the multiplicand) the same operation that was performed upon unity to produce a second number (called the multiplier). The result is called the product.

Ex. $3 \cdot 5 = 5+5+5$

The number 5 is taken as many times as 1 was taken to get 3.

(*Note*) It is evident that multiplication is shortened addition.

(2) Algebraic multiplication follows arithmetical multiplication directly. Owing to the enlargement of the number system by the use of the letter symbols and negative numbers, certain revisements to the arithmetical rules are necessary.

a Letter symbols may be multiplied in any order.
Ex. $ab = ba$

b Letter symbols may be grouped in any way
Ex. $xyz = x(yz) = y(xz) = z(xy)$

c Polynomial expressions may be multiplied by monomials by multiplying each term of the polynomial by the monomial.

Ex. $(x+y-z) \cdot a = ax+ay-az$

d Either a negative multiplicand or a negative multiplier will give a negative product.

Ex. $+a \cdot -b = -ab$

Ex. $-a \cdot +b = -ab$

e If both multiplicand and multiplier are either positive or negative, the product is positive.

Ex. $a \cdot b = +ab$

Ex. $-a \cdot -b = +ab$

(3) Exponents from definition indicate the number of times a number is taken as a factor, so the law follows that to multiply like factors, exponents are added.

Ex. $2^3 \cdot 2^4 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

Or, $2^3 \cdot 2^4 = 2^7$

Ex. $a^4 \cdot a^2 = a \cdot a \cdot a \cdot a \cdot a \cdot a$

Or $a^4 \cdot a^2 = a^6$

Multiplication of monomials.

(4) Monomials are multiplied by multiplying the coefficients together for a new coefficient, and the unknown factors together for a new unknown factor.

Ex. $3a \cdot 2b = 6ab$

Ex. $3xy \cdot 4xz = 12x^2yz$

Ex. $6c^2 \cdot 4d = 24c^2d$

Ex. $2a^n \cdot 3a^m = 6a^{n+m}$ (adding exponents)

(5) To multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial.

Ex. $3x \cdot (2xy + 3y - 4xz) = 6x^2y + 9xy - 12x^2z$

Ex. $2a(4b - c + 2d^2) = 8ab - 2ac + 4ad^2$

Ex. $6c^3(4d - 5cd + 3e) = 24c^3d - 30c^4d + 18c^3e$

Ex. $2(a^n-b^n)=2a^n-2b^m$

(*Note*) The laws of signs are given in working form as follows:

1 Like signs give a positive product.

2 Unlike signs give a negative product.

Problems

1 Multiply $5x$ by 2

2 Multiply $6ab$ by $3a$

3 Multiply $10x^2y$ by $2xy$

4 Multiply $12a^x$ by $4a^y$ (Adding exponents by indicating operation)

5 Multiply $2x+y$ by $3x$

6 Multiply $2x+4$ by $5x$

7 Multiply $4x^2+3x+4$ by $7x$

8 Multiply $6a^2-4a-2$ by $6a$

9 Multiply $-2x+4$ by $-3x$

10 Multiply $16x^2+4x-5$ by $-2x$

Lesson 8

Multiplication of polynomials by polynomials.

(6) To multiply a polynomial by a polynomial, multiply each term of the multiplicand by each term of the multiplier. Then arrange and add the several products.

Ex. Multiply $3x+y$ by $3x-y$

$$\begin{array}{ll} & 3x+y \\ & 3x-y \\ \hline \text{Multiplying by } 3x, & 9x^2+3xy \\ \text{Multiplying by } -y, & \quad -3xy-y^2 \\ \hline \text{Adding,} & 9x^2-y^2 \end{array}$$

Ex. Multiply $3x^2+2x-6$ by $2x+4$

$$\begin{array}{ll} & 3x^2+2x-6 \\ & 2x\ +4 \\ \hline \text{Multiplying by } 2x, & 6x^3+4x^2-12x \\ \text{Multiplying by } 4, & \quad +12x^2+8x-24 \\ \hline \text{Adding,} & 6x^3+16x^2-4x-24 \end{array}$$

Ex. Multiply $5cd-4$ by $2c+2$

$$\begin{array}{ll} & 5cd-4 \\ & 2c\ +2 \\ \hline \text{Multiplying by } 2c, & 10c^2d-8c \\ \text{Multiplying by } 2, & \quad +10cd-8 \\ \hline \text{Adding,} & 10c^2d-8c+10cd-8 \end{array}$$

(7) In multiplication, it is always advisable to ar-

range the terms of multiplicand and multiplier so that the exponents of the letter symbols under discussion are in ascending or descending order.

Ex. Multiply $3x^4+2x^2-3x^3+2-x$ by $3x+2x^2-3x^3$

Arranging terms,

$$\begin{array}{r} 3x^4-3x^3+2x^2-x+2 \\ -3x^3+2x^2+3x \\ \hline -9x^7+9x^6-6x^5+3x^4-6x^3 \\ +6x^6-6x^5+4x^4-2x^3+4x^2 \\ +9x^5-9x^4+6x^3-3x^2+6x \\ \hline \text{Adding, } -9x^7+15x^6-3x^5-2x^4-2x^3+x^2+6x \end{array}$$

(*Note*) The method of arrangement makes possible the writing of the products in columns, thus simplifying the process.

Definition—The *degree* of a *term* is determined by adding the exponents of the literal factors. The *degree* of an *expression* is determined by the term of highest degree in the expression.

Definition—An expression is homogeneous when the terms are all of the same degree.

Problems

1 Multiply $2a+1$ by $3a$
2 Multiply $6x^2+3x+2$ by $2x$
3 Multiply $6b+3c$ by $4b$
4 Multiply $15m^2n-2mn$ by mn
5 Multiply $6ax^2-3a^2x+4$ by $8x+2$
6 Multiply $5cd-3d^2+5$ by $2c-3d+2$

7 Multiply $10mn-4m^2+2n$ by $15m$

8 Multiply $3x^2y^3+5x^3y^2$ by $6x^4y^3$

9 Multiply $25ab^2-6a^2b$ by $5ab+3$

10 Multiply $14xy+7x^2-3y$ by $2xy-4x^2+y^2$

11 Expand $5x(2x^2+4y+3y^2)$

12 Expand $(3a+b)\ (6a^2b+10)$

13 Expand $(5x+y+z)\ (3x^2+4y-3z)$

14 Expand $(10x^3+4x^2-4x+2)\ (6x^2+3x-2)$

15 Expand $(6ab^3+4ab^2-3ab+6a)\ (2ab)$

Lesson 9

Literal exponents

1 The general rule for multiplication holds for expressions having literal exponents.

Ex. $(3a^{2p}+2a^{p}+a)\ (2a^{p}+1)$

$$\begin{array}{r} 3a^{2p}+2a^{p}+a \\ 2a^{p}+1 \\ \hline 6a^{3p}+4a^{2p}+2a^{p+1} \\ \quad +3a^{2p} \qquad\qquad +2a^{p}+a \\ \hline 6a^{3p}+7a^{2p}+2a^{p+1} \quad +2a^{p}+a \end{array}$$

It is essential to arrange terms in ascending or descending powers to get the best results.

Problems

1 $2x^{a}+3y^{2}-4x^{b}$
$4x^{a+1}+3y$

2 $6a^{m}-4b^{n}-3c^{2m}$
$3a^{2}+4b^{m}$

3 $8x^{2p}-4x^{p}+3$
$3x^{p}-x$

4 $15x^{2}y^{a}-4z^{m}$
$3xy-z^{n}$

DIVISION

Lesson 10

(1) If the product and one of two factors are given, it is possible to determine the other factor. This process is called division. The product is the dividend, the given factor is the divisor, and the factor found is the quotient.

(2) Division is expressed either by using the sign of division ($\div$), or by writing the dividend and divisor inthe form of a fraction, the dividend as the numerator the divisor as the denominator.

Ex. Divide $3ax$ by $2y$

Expressed as $3ax \div 2y$

Or $\dfrac{3ax}{2y}$

(3) To divide terms having like factors, subtract the exponents.

Ex. Divide $12x^3$ by $3x^2$

$$\frac{12x^3}{3x^2} = \frac{12 \cdot x \cdot x \cdot x}{3 \cdot x \cdot x} = 4x$$

(*Note*) The coefficients are divided to determine a new coefficient, and the given factors are divided to determine a new literal factor.

Ex. Divide $16x^3y^2$ by $2x^2y$

$$\frac{16x^3y^2}{2x^2y} = 8xy$$

(4) The law of signs is the same as for multiplication.

Like signs give a positive result.
Unlike signs give a negative result.

Ex. $-2a^2 \div -a = +2a$
$+22x^3 \div +11x^2 = +2x$
$16x^3y \div -2x^2 = -8xy$
$-8c^3d^2 \div +4cd^2 = -2c^2 \quad (d^2 \div d^2 = 1)$

Problems

1 Divide $15a^4$ by $5a^2$
2 Divide $10x^6$ by $2x$
3 Divide $20x^4y^2$ by $-10xy^2$
4 Divide $12m^6$ by $4m^3$
5 Divide $-6a^2b^2$ by $-2ab$

Lesson 11

Polynomials

(5) Polynomials are divided by monomials by dividing each term of the polynomial by the monomial.

Ex. $2x^4+4x^3-6x^2+4x$ is to be divided by $2x$

$$2x)\overline{2x^4+4x^3-6x^2+4x}$$
$$x^3+2x^2-3x+2$$

(6) In case the exponent of the divisor is larger than the exponent of the dividend, the new exponent is expressed by the negative number.

Ex. Divide $3a^4-2a^3+4a^2$ by $2a^4$

$$2a^4)\overline{3a^4-2a^3+4a^2}$$
$$\tfrac{3}{2}-a^{-1}+2a^{-2}$$

(*Note*) The negative exponent will be discussed at length in a following topic.

(7) Polynomials may be divided by polynomials as follows.

Divide $6x^6+7x^5+6x^4+4x^3+x^2$ by $2x^2+x$

$$)\overline{3x^4+2x^3+2^2+1}$$
$$6x^6+7x^5+6x^4+4x^3+x^2)\overline{2x^2+x}$$

Multiplying $(2x^2+x)$ by $3x^4$ $\quad 6x^6+3x^5$

Subtracting $\quad 4x^5+6x^4+4x^3+x^2$

Multiplying $(2x^2+x)$ by $2x^3$ $\quad 4x^5+2x^4$

Subtracting $\quad 4x^4+4x^3+x^2$

Multiplying $(2x^2+x)$ by $2x^2$ $\quad 4x^4+2x^3$

Subtracting $\quad 2x^3+x^2$

Multiylying $(2x^2+x)$ by 1 $\quad 2x^3+x^2$

Subtracting

Divide the first term of the dividend by the first term of the divisor.

$2x^2$ is contained $3x^4$ times in $6x^6$

$3x^4$ is the first term of the quotient.

Other terms of the quotient are determined in a similar manner.

Illustrative examples.

Divide $2a^2+7a+6$ by $a+2$

$$\begin{array}{rl} &)2a+3 \\ 2a^2+7a+6) & \overline{a+2} \\ \text{Multiplying } (a+2) \text{ by } (2a), & \underline{2a^2+4a} \\ \text{Subtracting,} & 3a+6 \\ \text{Multiplying } (a+2 \text{ by } (3), & \underline{3a+6} \qquad \text{quotient} = 2a+3 \end{array}$$

Divide $3x^4-x^3+2x^2-4x$ by $x-1$

$$\begin{array}{rl} &)3x^3+2x^2+4x \\ 3x^4-x^3+2x^2-4x) & \overline{x-1} \\ \text{Multiplying } (x-1) \text{ by } (3x^3), & \underline{3x^4-3x^3} \\ \text{Subtracting,} & 2x^3+2x^2-4x \\ \text{Multiplying } (x-1) \text{ by } (2x^2), & \underline{2x^3-2x^2} \\ \text{Subtracting,} & 4x^2-4x \\ \text{Multiplying } (x-1) \text{ by } (4x), & \underline{4x^2-4x} \\ & \text{quotient} = 3x^3+2x^2+4x \end{array}$$

Problems

1 Divide $4a^6+6a^4-4a^3+2a^2$ by $2a^2$

2 Divide $36a^6-27a^4$ by $9a^2$

3 Divide $a^2b+ab^2-4a^3b^3$ by ab

4 Divide a^2+3a+2 by $a+1$

5 Divide $6w^2-7w-3$ by $2w-3$

6 Divide $2x^4-6x^3+3x^2-5x+2$ by x^2-3x+1

7 Divide $6a^2-31ab+35b^2$ by $2a-7b$

8 Divide $2y^3-9y^2+11y-3$ by $2y-3$

9 Divide $a^2+b^2+1+2ab+2a+2b$ by $a+b+1$

10 Divide $x^2+y^2+z^2-2xy+2xz-2yz$ by $x-y+z$

Special Rules for Multiplication

Lesson 12

Owing to the fact that certain products recur time after time, it is of advantage to memorize certain short methods for multiplication.

(1) To multiply a binomial by itself.

Given the problem $(a+b)^2$

By multiplication

$$\begin{array}{r} a+b \\ \underline{a+b} \\ a^2+ab \\ \quad +ab+b^2 \\ \hline a^2+2ab+b^2 \end{array}$$

This product is made up of the squares of the two terms of the binomial plus twice the product of the two terms.

It is found that to square any binomial the product is always equal to the squares of the two terms plus twice the product of the two terms.

Ex. $(a-b)^2=a^2-2ab+b^2$

Ex. $(x+y)^2=x^2+2xy+y^2$

Ex. $(2a-b)^2=4a^2-4ab+b^2$

Problems

1	$(x-y)^2$	7	$(a-3b)^2$
2	$(c+d)^2$	8	$(a-b^2)^2$
3	$(m+n)^2$	9	$(2x^2-4y^3)^2$
4	$(2x+y)^2$	10	$(a+(a-b))^2$
5	$(x-2y)^2$	11	$(1+2x)^2$
6	$(2a+3b)^2$	12	$(x^2+3a)^2$

(2) To multiply the sum of two terms by their difference.

Given the problem $(a+b)\ (a-b)$

By multiplication

$$\begin{array}{r} a+b \\ a-b \\ \hline a^2+ab \\ -ab-b^2 \\ \hline a^2-b^2 \end{array}$$

It is noticed that the product is made up of the difference of the squares of the two terms. To multiply the sum of any two terms by their difference, the product is found always to equal the difference of the squares of the two terms.

Ex. $(x+y)\ (x-y)=x^2-y^2$

Ex. $(x+3y)\ (x-3y)=x^2-9y^2$

Problems

1	$(a-b)\ (a-b)$	7	$(1+2a)\ (1-2a)$
2	$(x+y)\ (x-y)$	8	$(6+x)\ (6-x)$
3	$(m+n)\ (m-n)$	9	$(2a^2+2)\ (2a^2-2)$
4	$(1+x)\ (1-x)$	10	$(4c-d)\ (4c+d)$
5	$(2a-b)\ (2a+b)$	11	$(2x^2-12)\ (2x^2+12)$
6	$(2x-4y)\ (2x+4y)$	12	$(1-3a^2)\ (1+3a^2)$

Lesson 14

(3) To multiply two binomials together which have a common term.

Given the problem $(x+2)\ (x+3)$

By multiplication,

$$\begin{array}{r} x+2 \\ x+3 \\ \hline x^2+2x \\ \quad +3x+6 \\ \hline x^2+5x+6 \end{array}$$

The product is made up of the square of the common term, plus the common term with a coefficient equal to the sum of the unlike terms, plus the product of the unlike terms.

Ex. $(a+4)\ (a-2)=a^2+2a-8$

Ex. $(c+7)\ (c+2)=c^2+9a+14$

Problems

1	$(a+2)\ (a+3)$	7	$(x+6)\ (x-6)$
2	$(x-4)\ (x+2)$	8	$(2a-3)\ (2a+4)$
3	$(m+2)\ (m-1)$	9	$(x-2)\ (x+10)$
4	$(x+3)\ (x+5)$	10	$(y-6)\ (y+5)$
5	$(y-7)\ (y-3)$	11	$(a^2+2)\ (a^2-5)$
6	$(r-2)\ (r+5)$	12	$(x+11)\ (x-8)$

(4) Any two binomials may be quickly multiplied together by, (1) multiplying the first terms of the binomials together, (2) multiplying the second terms of the binomials together, and (3) multiplying the terms together in such a manner as will give cross products.

Ex. $(2a+4)\,(3a-2)$

Multiplying first terms together, $6a^2$

Multiplying second terms together, -8

The cross products are, $12a$ and $-4a$

Therefore product equals $6a^2+8a-8$

(*Note*) The cross products may be easily determined by the following process.

$$\overbrace{(2a+4)\ (3a-2)}$$

Ex. $(6c-2)\,(2c+3)=12c^2+14c-6$

$12c^2$ = product of first terms

-6 = product of second terms

$14c$ = sum of cross products

Problems

1	$(a+2)\,(a+3)$	7	$(6-3y)\,(4-4y)$
2	$(x+y)\,(w-z)$	8	$(2-x)\,(3-y)$
3	$(2a-3)\,(4a+4)$	9	$(4b+2)\,(3-2b)$
4	$(6-3x)\,(4+4y)$	10	$(a-4)\,(2a-2)$
5	$(5-2z)\,(2+4z)$	11	$(6+10x)\,(x-3)$
6	$(1+3a)\,(b+4a)$	12	$(a+4)\,(a-2b)$

Special Rules for Division

Lesson 16

The special rules for division are derived directly from the special rules for multiplication, and lead directly into factoring. Factoring will be considered as a special application of the short rules for division.

(1) If the squaring of a binomial gives a product made up of, (1) the square of the first term, (2) twice the product of the two terms, (3) the square of the second term, then any trinomial, whose terms are two squares and a third term equaling twice the product of the square roots of the squared terms, is divisible by a binomial made up of the square roots connected by the sign of the second term.

Ex. If $(a+b)^2=a^2+2ab+b^2$

Then $\dfrac{a^2+2ab+b^2}{a+b}=a+b$

Ex. If $(a-b)^2=a^2-2ab+b^2$

Then $\dfrac{a^2-2ab+b^2}{a-b}=a-b$

(*Note*) The terms $(a-b)$ and $(a-b)$ are factors of $a^2-2ab+b^2$

(2) A trinomial is a perfect square if two of the terms are squares and the third is equal to twice the square roots of the squared terms.

Ex. $a^2-2ab+b^2$ is a perfect square,

For, a^2 and b^2 are squares,

And $-2ab$ is twice the product of the square roots.

Ex. $4c^2+4cd+d^2$ is a perfect square,
For $4c^2$ and d^2 are perfect squares,
And $4cd$ is twice the product of the square roots.
Therefore $4c^2+4cd+d^2$ is divisible by $2c+d$.

Ex. $a^2x^2+2ax+1$ is a perfect square,
For a^2x^2 and 1 are perfect squares,
And $2ax$ is twice the product of the square roots.
Therefore $a^2x^2+2ax+1$ is divisible by $ax+1$

Problems

Determine which of the following are perfect squares

1	$x^2+2xy+y^2$	7	$4c^2+8cd+4d^2$
2	$a^2+4ab+4b^2$	8	$x^4+x^2y^2+y^4$
3	$x^2-2xy+y^2$	9	$9a^2-6ab+b^2$
4	x^2+xy+y^2	10	$1+2x+x^2$
5	$25a^2+10ab+b^2$	11	$16y^2+8y+1$
6	$y^2-21yz+z^2$	12	$36w^2-24w+4$

Lesson 17

If $(a+b)\ (a-b)=a^2-b^2$

Then $\dfrac{a^2-b^2}{a+b}=a-b$

And $\dfrac{a^2-b^2}{a-b}=a+b$

(3) The difference of two squares is always divisible by either the sum or the difference of the square roots.

Ex. c^2-d^2 is divisible by either $c+d$ or $c-d$

$$\frac{c^2-d^2}{c+d}=c-d$$

$$\frac{c^2-d^2}{c-d}=c+d$$

(*Note*) $(c+d)$ and $(c-d)$ are factors of c^2-d^2

Ex. $4a^2x^2-16c^2$ is divisible by either $(2ax+4c)$ or $(2ax-4c)$

$$\frac{4a^2x^2-16c^2}{2ax+4c}=2ax-4c$$

$$\frac{4a^2x^2-16c^2}{2ax-4c}=2ax+4c$$

(*Note*) $(2ax+4c)$ and $(2ax-4c)$ are factors of $4a^2x^2-16c^2$

Problems

1 Divide (x^2-y^2) by $(x-y)$

2 Divide a^4-b^4 by a^2+b^2

3 Divide x^2-9 by $x-3$

4 Divide x^2-9 by $x+3$
5 Divide $4x^4-16$ by $2x^2-4$
6 Divide $4a^2+4$ by $2a-2$
7 Divide $9-x^2$ by $3+x$
8 Divide $16a^2-b^2$ by $4a+b$
9 Divide a^2b^2-4 by $ab-2$
10 Divide $9x^6-c^2$ by $3x^3+c$
11 Divide $16-4a^2$ by $4+2a$
12 Divide a^2-b^2 by $a+b$
13 Divide $1-a^2$ by $1-a$
14 Divide $1-a^4$ by $1-a^2$

Lesson 18

It is found by actual division that

$$\frac{x^3+y^3}{x+y}=x^2-xy+y^2$$

And $$\frac{x^3-y^3}{x-y}=x^2+xy+y^2$$

It is noticed that the sum of two cubes may be divided by the sum of the first powers of the quantities, and that the difference of two cubes may be divided by the difference of the first powers of the quantities.

The quotients are so striking as to be easily remembered.

(4) Dividing the sum of the cubes of two quantities by the sum of the quantities, we get the sum of the squares of the quantities minus the product of the quantities.

(5) Dividing the difference of the cubes of two quantities by the difference of the first powers, we get the sum of the squares of the quantities plus the product of the quantities.

Ex. $$\frac{8a^3+b^3}{2a+b}=4a^2-2ab+b^2$$

$$\frac{8a^3-b^3}{2a-b}=4a^2+2ab+b^2$$

Ex. $\dfrac{27x^3+y^3}{3x+y}=9x^2-3xy+y^2$

$\dfrac{27x^3-y^3}{3x-y}=9x^2+3xy+y^2$

Ex. $\dfrac{w^3+z^3}{w+z}=w^2-wz+z^2$

$\dfrac{w^3-z^3}{w-z}=w^2+wz+z^2$

The sum or the difference of two cubes is, therefore, factorable, and with a short drill the factors may be determined by inspection.

Problems

1 Divide a^3-b^3 by $a-b$
2 Divide x^3+y^3 by $x+y$
3 Divide $8-x^3$ by $2-x$
4 Divide $27a^3-b^6$ by $3a-b^2$
5 Divide x^6-y^6 by x^2-y^2
6 Divide $64-x^3$ by $4-x$
7 Divide $1-x^3$ by $1-x$
8 Divide $1+y^3$ by $1+y$
9 Sivide $125+x^3$ by $5+x$
10 Divide y^3-125 by $y-5$
11 Divide $27x^3-8$ by $3x-2$
12 Divide $a^3b^3-d^3$ by $ab-d$

Factoring

Lesson 19

The direct application of the short methods of division is in the determination of the factors of an expression. Before considering the special cases of factoring it is advisable to memorize the following definitions.

(1) A *rational* expression is one containing no indicated root of the letter considered.

$2x^2+x+4$ is rational.

$2x^2+\sqrt{x}+4$ is irrational.

(*Note*) The sign $\sqrt{\ }$ is used to indicate a root to be found. The particular root is indicated by a number written in the $\sqrt{\ }$.

Ex. $\sqrt[2]{\ }$ or $\sqrt{\ }$ second root (square root).

Ex. $\sqrt[3]{\ }$ third root (cube root).

Ex. $\sqrt[4]{\ }$ fourth root.

If no number is written in the $\sqrt{\ }$ the root is understood to be the second.

(2) An integral expression is one in which the letter considered does not appear in a denominator.

Ex. $3y+6$ is integral.

$\frac{2}{y}+4y+8$ is not integral.

(3) The factors of an expression are the expressions multiplied together to produce it.

(4) It follows that the integral factors of an expression are the integral expressions multiplied together to produce it.

(5) Also the rational factors of an expression are the rational expressions multiplied together to produce it.

Problems

1 Distinguish between the words rational and irrational.

2 What sign is used to indicate that a root is to be taken?

3 Define integral expression.

4 Define factor.

Lesson 20

(6) To factor trinomials which are perfect squares.

Ex. $a^2+2ab+b^2$ (formed by squaring $(a+b)$)
Factors are $(a+b)$ $(a+b)$

This follows from the special rule for multiplication.

(7) To factor the difference of two squares.

Ex. x^2-y^2. (Product of $(x+y)$ and $(x-y)$)
Factors are $(x+y)$ $(x-y)$

Problems

Factor—

1 $x^2-2xy+y^2$
2 $a^2+2ab+b^2$
3 $m^2-2mn+n^2$
4 $4x^2-4x+1$
5 $36a^2b^2-24abc+4c^2$
6 $16x^2-32xy+16y^2$
7 Factor the problems in Lesson 16.

Factor—

1 (x^2-y^2)
2 (a^4-b^4)
3 (x^4-y^4)
4 $(4x^2-b^2)$
5 $(9y^2-1)$
6 $25a^2-16b^2$
7 $36a^2b^2-4$
8 $25-4b^2$
9 $9x^2-16y^4$
10 $4w^2-9$

Lesson 21

(8) To factor the sum of two cubes.

Ex. x^3+y^3 (Product of $(x+y)$ and (x^2-xy+y^2))

Factors are $(x+y)$ (x^2-xy+y^2)

(See special rule for division.)

(9) To factor the difference of two cubes.

Ex. x^3-y^3 (Product of $(x-y)$ and (x^2+xy+y^2))

Factors are $(x-y)$ (x^2+xy+y^2)

(See special rule for division.)

1	a^3+b^3	7	a^3-b^3
2	x^3-y^3	8	$1-w^3$
3	a^3-c^3	9	a^3b^3-8
4	$8x^3+1$	10	$1-c^6$
5	$27x^6-y^3$	11	a^9-64
6	$1+x^6$	12	a^3-1

Lesson 22

(10) To factor trinomials made up of two factors having a common term.

Ex. x^2-x-6. (Product of $(x-3)$ and $(x+2)$)
Factors are $(x-3)\ (x+2)$

The method is to determine two factors of the constant (in this case -6) whose sum is the coefficient of the first power term (in this case -1).

Ex. x^2+5x+6. (Product of $(x+3)$ and $(x+2)$)
Factors are $(x+3)\ (x+2)$
Factors of 6, whose sum is 5, are 2 and 3.

Ex. a^2-5x+6. (Product of $(a-2)$ and $(a-3)$)
Factors are $(a-2)\ (a-3)$
Factors of 6, whose sum is -5, are -3 and -2.

Problems

1	x^2+x-6	7	$a^2x^2-6ax+5$
2	x^2+5x+6	8	$4+4x+x^2$
3	$x^2+7x+10$	9	$a^2+12+20$
4	$a^2-3x-10$	10	$1-2x+x^2$
5	$y^2+15x+26$	11	g^2+4g+4
6	$w^2-4wy+4y^2$	12	$a^2+10a+16$

Lesson 23

(11) In case the terms of an expression have a common divisor, this may be removed and considered as a factor.

Ex. $2a+4a^2x+6ay$

$2a$ is common divisor.

Dividing, $2a(1+2ax+3y)$

The factors of the expression are $2a$ and $1+2ax+3y$

Ex. $6ayw-2za+4a^2y^3w$

$2a$ is common divisor.

Dividing, $2a(3yw-z+2ay^3w)$

The factors of the expression are $2a$ and $3yw-z+2ay^3w$

Problems

1	$ax+ay$	8	$2x^2y-4x^2y$
2	$2b^2-b$	9	a^3+4a^2-10a
3	$3c+6c^2-9c^3$	10	$3xyz+4x^3y^3-5y^2z^3$
4	$27a^2-9$	11	$5c^2+10c$
5	$3xy+x^2y^2$	12	$15y^3+6y^2+3y$
6	$2w^3-w^2$	13	$(a+b)^2+3(a+b)$
7	$4+5a+8$	14	$4(x-y)^4-2(x-y)^2$

Lesson 24

(12) If one considers that the sign of aggregation is used to show that two or more terms are to be considered as one term, expressions involving binomial terms may be factored.

Ex. $(a+b)^2+2(a+b)+1$ may be divided into the factors $((a+b)+1)\ ((a+b)+1)$

(*Note*) $(a+b)$ is considered as a single term.

a^2+2a+1
$(a+b)^2+2(a+b)+1$ } These expressions are in the same form.

Ex. $(m+2n)^2+2(m+2n)\ (x-y)+(x-y)^2$

Factors are, $[(m+2n)+(x-y)]$ and $[(m+2n)+(x-y)]$

(*Note*) In this problem $(m+2n)$ and $(x-y)$ are considered as single terms.

$a^2+2ab+b^2$
$(m+2n)^2+2(m+2n)\ (x-y)+(x-y)^2$ } These expressions are in the same form.

Ex. $(a+b)^2-(x+y)^2$

Factors are $[(a+b)+(x+y)]$ and $(a+b)-(x+y)]$

a^2-b^2
$(a+b)^2-(x+y)^2$ } These expressions are in the same form.

Ex. $(x+y)^2+5(x+y)+6$

Factors are $[(x+y)+3)]$ and $[(x+y)+2)]$

a^2+5a+6
$(x+y)^2+5(x+y)+6$ } These expressions are in the same form.

Ex. $2(a-2b)+4x(a-2b)-6(a-2b)$

Factors are $(a-2b)$ and $(2+4x-6)$ (Taking out the common factor.)

$2a+4xa-6a$ } These expressions
$2(a+2b)+4x(a-2b)-6(a-2b)$ } are in the same form.

Problems

1 $x^2+2x(a+b)+(a+b)^2$
2 $(a-b)^2-2(a-b)\ (a+b)+(a+b)^2$
3 $(x-y)^2-(x+y)^2$
4 $(2a-b)^2-(x-y)^2$
5 $(4b+2)^2-4$
6 $(a-b)^2+2(a-b)+1$
7 $(a+b)^2+5(a+b)+6$
8 $(x-y)^2-(x-y)-6$
9 $(c+d)^2+7(c-d)+10$
10 $3(a-b)+4(a-b)+2(a-b)$
11 $4(a-b)^2+2(a-b)$
12 $6(x+y)^2+9(x+y)^4$

Lesson 25

(13) Expressions may be grouped in many cases so that they are factorable.

Ex. $ax+ay+bx+by$

By grouping the first two terms and the last two terms,

$(ax+ay)+(bx+by)$

Factoring each group, $a(x+y)+b(x+y)$

Considering this expression as made up of two terms, $(x+y)$ may be taken out as a common factor.

$(x+y)\ (a+b)$ The factors.

Ex. $2xy+3axy-6xw-9axw$

Grouping the first and third terms and the second and fourth terms,

$(2xy-6xw)+(3axy-9axw)$

Factoring each group $2x(y-3w)+3ax(y-3w)$

Considering this expression as made up of two terms, it is factorable.

$(y-3w)\ (2x+3ax)$. The factors.

Problems

1 $ax+bx+ay+by$

2 $2x-3x+2y-3y$

3 $4a+6b+8a+12b$

4 $6x-12+18x^2-36x$

5 $3ax+2a+9ax^2+6ax$

6 $x^3-x^2y+xy^2-y^3$

7 $3a^3-6ya^2-a+2y$

8 $mn-np-m+p$

9 $a^2+3b-3a-ab$

10 $a+2b-4a-8b$

Lesson 26

(14) Often an expression may be grouped to give the difference of a trinomial square term and a monomial square term.

Ex. $a^2+2ab+b^2-x^2$

Grouping, $(a^2+2ab+b^2)-x^2$

Or $(a+b)^2-x^2$

Considering $(a+b)$ as a single term, this is the difference of two squares.

Factoring, $[(a+b)+x)]\ [(a+b)-x)]$,

Or $(a+b+x)\ (a+b-x)$

Ex. $a^2-2ab+b^2-16$

Grouping, $(a^2-2ab+b^2)-16$

Or $(a-b)^2-16$

Factoring, $[(a-b)+4)]\,[(a-b)-4)]$

Or $(a-b+4)\ (a-b-4)$

Problems

1 $x^2+2xy+y^2-a^2$

2 $a^2-2ab+b^2-c^2$

3 $4x^2+4x+1-x^4$

4 $c^2-(a+b)^2$

5 $x^2-a^2-2ab-b^2$

6 $a^2-2a+1-16m^4$

7 $a^2+2ab+b^2-x^2-2xy-y^2$

8 $m^2-2mn+n^2-r^2-2rs-s^2$

9 $4m^4n^2-c^2-4cd-4d^2$

10 $(a-b)^2-x^2-2xy-y^2$

11 $1-x^2-2x-1$

12 $x^2+2xy+y^2-49$

Highest Common Factor
Lowest Common Multiple
Lesson 27

By inspection it is possible to determine the highest common factor of two or more expressions.

Ex. $2ax+2xy$. Factors are $2x(a+y)$

$a^2+2ay+y^2$. Factors are $(a+y)\ (a+y)$

The highest common factor is $(a+y)$

(*Note*) To determine the highest common factor it is necessary first to factor the expressions under consideration.

(15) The lowest common multiple of two or more expressions is the lowest expression which will contain the expressions. It is determined first by factoring the expressions and then taking each factor the greatest number of times it appears in an expression.

Ex. x^2-y^2 Factors are $(x+y)\ (x-y)$

$x^2+2xy+y^2$ Factors are $(x+y)\ (x+y)$

$2wx+2wy$ Factors are $2w(x+y)$

$2w(x+y)\ (x-y)\ (x+y)$ will contain the three expressions.

(16) The working method is to take all the factors of the first expression and all the factors of the other expressions not already selected.

L. C. M. $=2w(x+y)\ (x+y)\ (x-y)$

Ex. $2ax+4a^2y-6ab$ Factors are $2a(x+2ay-3b)$

$4x^2-y^2$ Factors are $(2x-y)\ (2x+y)$

$x^2+7x+10$ Factors are $(x+5)\ (x+2)$

$4x^2+4xy+y^2$ Factors are $(2x+y)\ (2x+y)$

x^2-x-6 Factors are $(x-3)\ (x+2)$

L. C. M. $=2a(x+2ay-3b)\ (2x-y)\ (2x+y)^2\ (x+5)\ (x+2)\ (x-3)$

Problems

Determine the H.C.F and L.C.M. of the following.

1 $\begin{cases} 2x \\ 4x^2y \\ 6xy^2 \end{cases}$

2 $\begin{cases} 24ax^2 \\ 16a^2x \\ 4a^3x^3y \end{cases}$

3 $\begin{cases} a^2-b^2 \\ a^2+2ab+b^2 \end{cases}$

4 $\begin{cases} a^2+2ab+b^2 \\ a^3-b^3 \end{cases}$

5 $\begin{cases} x^2+5x+6 \\ x^2+6x+9 \end{cases}$

6 $\begin{cases} a^3+b^3 \\ a^2-b^2 \end{cases}$

7 $\begin{cases} ax+ay \\ x^2+2xy+y^2 \end{cases}$

8 $\begin{cases} x^2y-xy^2 \\ x^2-2xy+y^2 \\ x^2-y^2 \end{cases}$

9 $\begin{cases} a-2 \\ a^2+x-6 \\ a^2-4a+4 \\ a^3-8 \end{cases}$

10 $\begin{cases} 2x^2-2x-6 \\ (x-3)^2 \\ x^3-27 \end{cases}$

SECTION III

FRACTIONS

Lesson 1

General definitions.

1 An algebraic fraction is the indicated quotient of two expressions.

2 The expressions are the *terms* of the fraction.

3 The numerator is the dividend, the denominator is the divisor.

Fundamental laws.

(1) The terms of a fraction may be multiplied or divided by the same quantity without changing the value of the fraction.

Ex. If $\frac{x}{y}=w$. then $\frac{ax}{ay}=w$

This is evident when one considers that the (a) factor of the numerator contains the factor (a) of the denominator once.

Ex. If $\frac{x}{y}=w$, then $\frac{\frac{x}{a}}{\frac{y}{a}}=w$

If $\frac{y}{a}$ is considered the divisor, then dividing $\frac{x}{a}$ by $\frac{y}{a}$ is equivalent to multiplying $\frac{x}{a}$ by $\frac{a}{y}$

$$\text{Or } \frac{x}{a}\cdot\frac{a}{y}=\frac{xa}{ya}$$

From the preceding law for multiplication,

$$\frac{xa}{ya}=\frac{x}{y}=w$$

(2) By application of the laws for signs in division.

$a \div b$ gives a $(+)$ sign.
$-a \div -b$ gives a $(+)$ sign
$-a \div b$ gives a $(-)$ sign
$a \div -b$ gives a $(-)$ sign

Or $\frac{a}{b}$ gives $+$, $\frac{-a}{-b}$ gives $+$, $\frac{-a}{b}$ gives $-$, $\frac{a}{-b}$ gives $-$.

It follows that if the quotient of $\frac{-a}{b}$ gives $-$, that $-\frac{-a}{b}$ gives $+$, and that $-\frac{a}{-b}$ gives $+$.

$$\text{Or } \frac{a}{b}=\frac{-a}{-b}=-\frac{-a}{b}=-\frac{a}{-b}$$

(3) From the laws for removal of signs of aggregation preceded by the negative sign,

$$-(x+y)=-x-y,$$
$$\text{Or } -(x-y)=-x+y=y-x,$$
$$\text{Or } -(-x+y)=x-y.,$$

It follows that if a binomial factor appears in either

the numerator or the denominator of a fraction, changing the signs of the terms of the binomial changes the sign of the fraction.

$$Ex. \frac{a}{x-y} = \frac{-a}{y-x}$$

$$Ex. \frac{a-b}{a+b} = -\frac{b-a}{a+b}$$

Problems

1 Define fraction, terms of a fraction, numerator, denominator.

2 Discuss the fundamental laws governing the fraction.

3 What laws of signs hold for the fraction.

REDUCTION

Lesson 2

Reduction of fractions to equivalent forms.

(1) Applying the law for division of both terms of a fraction by the same quantity.

$$\frac{6a^2b^3}{3abc} = \frac{2.3.a.a.b.b.b}{3.a.b.c} = \frac{2ab^2}{c}$$

In the above problem, we divide both terms by the quantity $3ab$, giving $\frac{2ab^2}{c}$

It follows that to reduce a fraction to its lowest terms, it is only necessary to divide both terms of the fraction by the highest common factor of the terms.

$$Ex. \quad \frac{42a^2b^3c}{28ab^4c^2} = \frac{6a}{4bc}$$

(Dividing by $7ab^3c$)

$$Ex. \quad \frac{50(c+d)^2}{20(c+d)} = \frac{5(c+d)}{2}$$

(Dividing by $10(c+d)$

If the terms of a fraction are made up of polynomials, (1) factor both terms, (2) divide by the highest common factor.

$$Ex. \quad \frac{x^4+2x^2+1}{x^4-1} = \frac{(x^2+1)^2}{(x^2-1)(x^2+1)} = \frac{x^2+1}{x^2-1}$$

(Dividing by (x^2+1)

$$Ex. \quad \frac{x^3+1}{x^2+2x+1}=\frac{(x+1)\,(x^2-x+1)}{(x+1)^2}=\frac{x^2-x+1}{x+1}$$

(Dividing by $(x+1)$)

$$Ex. \quad \frac{3x+3b}{x^2+2ab+b^2}=\frac{3(x+b)}{(x+b)^2}=\frac{3}{x+b}$$

(Dividing by $(x+b)$)

(2) To reduce a fraction to an integral or mixed expression.

$$Ex. \ \frac{3ab}{ab}=3 \qquad Ex. \ \frac{x^2+2xy+y^2}{x+y}=\frac{(x+y)^2}{x+y}=x+y$$

$$Ex. \ \frac{6x^2y}{3x}=2xy \quad Ex. \ \frac{(a-b)^3}{a-b}=(a-b)^2$$

In the above the denominator is contained an intergal number of times in the numerator.

$$Ex. \quad \frac{6x+y}{2x}=\frac{6x}{2x}+\frac{y}{2x}=3+\frac{y}{2x}$$

$$Ex. \quad \frac{a^4+1}{a+1}$$

Dividing the numerator by the demoninator.

$$\begin{array}{r|l}
a^4+1 & a+1 \\
\cline{2-2}
a^4+a^3 & a^3-a^2+a \\
\cline{1-1}
-a^3+1 & \\
-a^3-a^2 & \\
\cline{1-1}
+a^2+1 & \\
+a^2+a & \\
\cline{1-1}
-a+1 &
\end{array}$$

The quotient is a^3-a^2+a with a remainder of $-a+1$. This remainder is placed over the divisor and the fraction is added to the quotient.

$$a^3-a^2+a+\frac{-a+1}{a+1}$$

To reduce a fraction to an integral or mixed expression, it is only necessary to divide the numerator by the denominator, and in case of a remainder to place it over the divisor and add the resulting fraction to the quotient already found.

Problems

Reduce to lowest terms

1 $\frac{6ab}{2a}$

2 $\frac{12x^2y}{6xy}$

3 $\frac{18a^2bx}{3ax}$

4 $\frac{2a}{4a}$

5 $\frac{a^2-b^2}{a^2-2ab+b^2}$

6 $\frac{x^2+5x+6}{x^2+6x+9}$

7 $\frac{4x+3}{2x}$

8 $\frac{6a^2+4a+5}{2a}$

9 $\frac{4x^3+3x^2+x+1}{x+1}$

10 $\frac{5a^2+b-c}{5a^2+b}$

Addition and Subtraction of Fractions

Lesson 3

Addition and subtraction of fractions.

(1) To add or subtract fractions it is necessary that they have the same denominator.

$$Ex. \quad \frac{x}{y}+\frac{3x}{y}+\frac{4x}{y}=\frac{8x}{y}$$

This is evident when one considers that

$$\frac{x+y+w}{z}=\frac{x}{z}+\frac{y}{z}+\frac{w}{z}$$

(2) Fractions which do not have the like denominators may often be written as equivalent fractions which have the like denominators. They may then be added.

$$Ex. \quad \frac{x}{3}+\frac{x}{4}=\frac{4x}{12}+\frac{3x}{12}=\frac{7x}{12}$$

In the above problem both members of $(\frac{x}{3})$ were multiplied by (4), and both members of $(\frac{x}{4})$ were multiplied by (3). The resulting fractions have the same denominator (12), and may be added.

To combine two or more fractions so that they may have the one denominator is called *simplification*.

$$Ex. \quad \text{Simplify } \frac{x}{a}+\frac{y}{b}-\frac{w}{c}$$

$$\frac{x}{a}=\frac{xbc}{abc},\quad \frac{y}{b}=\frac{yac}{abc},\quad -\frac{w}{c}=-\frac{wab}{abc}$$

Combining $\dfrac{xbc+yac-wab}{abc}$

Ex. $\dfrac{a}{x}+\dfrac{b}{x(x+y)}$

$$\frac{a}{x}=\frac{a(x+y)}{x(x+y)},$$

Combining $\dfrac{a(x+y)+b}{x(x+y)}$

It is best first to determine the lowest common denominator of the denominators of the fractions to be combined. This prevents needless multiplication of terms.

Ex. $\dfrac{a}{(x+y)^2}-\dfrac{b}{x^2-y^2}$

L.C.D. of $(x+y)^2$ and x^2-y^2 is $(x+y)^2(x-y)$

$$\frac{a}{(x+y)^2}=\frac{a(x-y)}{(x+y)^2(x-y)},\quad -\frac{b}{x^2-y^2}=-\frac{b}{(x+y)(x-y)}$$

$$=-\frac{b(x+y)}{(x+y)^2(x-y)}$$

Combining, $\dfrac{a(x-y)-b(x+y)}{(x+y)^2(x-y)}$

(*Note*) Time will be saved if the denominators of the fractions are first factored. The lowest common denominator is then more easily determined.

Ex. $\dfrac{x+1}{2x+2}+\dfrac{x-1}{4x-4}-\dfrac{x+3}{x^2-1}$

Lowest common denominator.

$$\left.\begin{matrix} 2(x+1) \\ 4(x-1) \\ (x+1)\,(x-1) \end{matrix}\right\} \quad 4(x+1)\,(x-1)$$

Multiplying each fraction by such a number as will give equivalent fractions having the same denominators.

$$\frac{x+1}{2(x+1)} = \frac{2(x+1)\,(x-1)}{4(x+1)\,(x-1)} = \frac{2(x^2-1)}{4(x^2-1)}$$

$$\frac{x-1}{4(x-1)} = \frac{(x-1)\,(x+1)}{4(x+1)\,(x-1)} = \frac{x^2-1}{4(x^2-1)}$$

$$-\frac{x+3}{x^2-1} = -\frac{4(x+3)}{4(x+1)\,(x-1)} = -\frac{4(x+3)}{4(x^2-1)}$$

Combining, $\dfrac{2(x^2-1)+(x^2-1)-4(x+3)}{4(x^2-1)}$

Or, $\dfrac{3(x^2-1)-4(x+3)}{4(x^2-1)}$

(3) Often fractions may be combined more easily if the signs of the factors of the denominators are changed.

Ex. $\dfrac{2a}{(x+y)\,(x-y)} + \dfrac{2b}{(x+y)\,(y-x)}$

This may be written $\dfrac{2a}{(x+y)\,(x-y)} - \dfrac{2b}{(x+y)\,(x-y)}$

$$= \frac{2a-2b}{(x+y)(x-y)}$$

Problems

Add.

1 $\frac{2}{x}, \frac{3}{2x}$

2 $\frac{4a}{2}, \frac{3a}{3}, \frac{5a}{4}$

3 $\frac{6x,}{a^2-b^2} \quad \frac{4x}{a^2-2ab+b^2}$

4 $\frac{(a-b)}{a^2-2ab+b^2}, \quad \frac{a+b}{a-b}$

5 $\frac{3,}{2x}, \frac{5a}{4x^2+6x}$

6 $\frac{5a}{2}, \quad \frac{-6a}{3}, \quad \frac{4a}{4}$

7 $\frac{3(a-b)}{2x-y}, \quad \frac{2a+4b}{4x^2-4xy+y^2}$

8 $\frac{10,}{a-b}, \; -\frac{5}{a+b}$

Combine

9 $\frac{3}{ax} - \frac{2}{ax^2} + \frac{4}{x}$

10 $\frac{2}{a-b} - \frac{4}{a^2-b^2} + \frac{6}{a+b}$

11 $\frac{5x}{2x+1} + \frac{3x}{2} - \frac{4x}{2x-1}$

12 $\frac{x+1}{x^2-4x+4} - \frac{x-2}{x^2-4} - \frac{2}{x}$

13 $\dfrac{2a}{a^3-b^3} + \dfrac{4a}{a-b} - \dfrac{6a}{a^2+ab+b^2}$

14 $\dfrac{2}{x} - \dfrac{6}{3x} + \dfrac{4}{2x+4x^2}$

15 $\dfrac{a-b}{a^2+7x+10} - \dfrac{a+b}{a^2+8x+12}$

Multiplication of Fractions

Lesson 4

(1) As the numerator stands for the number of equal parts taken, it is evident that to multiply the numerator of a fraction by any number is equivalent to multiplying the fraction by that number.

$$Ex.\quad \frac{1}{2} \cdot 2 = \frac{2}{2} = 1$$

$$Ex.\quad \frac{2}{3} \cdot 3 = \frac{6}{3} = 2$$

$$Ex.\quad \frac{a}{b} \cdot a = \frac{a^2}{b}$$

(2) As the denominator of a fraction is the divisor, it is also evident that to multiply the denominator of a fraction by any number is equivalent to dividing the fraction by that number.

Ex. $\frac{1}{2}$ is divided by (2) by multiplying the denominator (2) by (2). ($\frac{1}{4}$) is one half of ($\frac{1}{2}$).

Ex. $\frac{3}{4}$ is divided by (2) by multiplying the denominator (4) by (2).

(3) To multiply a fraction by a fraction, it is only necessary to multiply the numerators together for a new numerator and the denominators together for a new denominator.

Ex. $\frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$

$\frac{3}{4}$ is to be taken $\frac{1}{2}$ times. This means that the fraction is to be divided by 2.

In the problem, $\frac{3}{4} \cdot \frac{2}{3}$

$\frac{3}{4}$ is taken $\frac{2}{3}$ times, which means that the fraction is first to be divided by (3) and then (2) parts taken. To divide $\frac{3}{4}$ by (3), multiply the denominator by (3). To take (2) of these parts, multiply the fraction by (2) by multiplying the numerator by (2).

$$Ex. \quad \frac{6a^2xy}{7a^3xz^2} \cdot 3xy = \frac{18x^2y^2a^2}{7a^3xz^2}$$

(Multiplying the numerator by $3xy$)

$$Ex. \quad \frac{6a^2b}{4c} \cdot \frac{8a^3b}{5a^2} \cdot \frac{2xab}{2ac} = \frac{96a^6b^3x}{40c^2a^3}$$

(Multiplying the numerators together for a new numerator, and the denominators together for a new denominator.)

$$Ex. \quad \frac{x+1}{2x-3} \cdot \frac{x-1}{6} = \frac{(x+1)(x-1)}{6(2x-3)} = \frac{x^2-1}{12x-18}$$

(Multiplying the numerators together for a new numerator, and the denominators together for a new denominator.)

Problems

1 Multiply $\frac{2}{x}$ by 3

2 Multiply $\frac{4a}{2x}$ by 5

3 Multiply $\frac{2y}{3}$ by 8

4 Multiply $\frac{a-b}{2}$ by $\frac{1}{2}$

5 Multiply $\frac{3}{2a}$ by $\frac{4}{2b}$

6 Multiply $\frac{a-b}{2}$ by $\frac{a}{a+b}$

7 Multiply $\frac{2x+y}{x-y}$ by $\frac{x+y}{2x}$

8 Multiply $\frac{1-x}{3}$ by $\frac{1}{1+x^2}$

9 Multiply $\frac{2}{3x}$ by $\frac{4}{6x}$

10 Multiply $\frac{1}{3y}$ by $\frac{2}{5y^2}$

11 Multiply $\frac{2x+2}{4x}$ by $3x$

12 Multiply $\frac{3x^2+2x-1}{x-2}$ by $\frac{3x}{x+3}$

13 Multiply $\frac{(a-b)^2}{2x+4}$ by $\frac{a-b}{2x}$

14 Multiply $\frac{(a-b)^2}{a-b}$ by $\frac{a-b}{a+ab+b}$

15 Multiply $\frac{2}{3y-1}$ by $\frac{4x-2}{4}$

16 Multiply $\frac{a^2+3x-10}{a^2+8x+15}$ by $\frac{(a+5)^2}{a-3}$

Note. Factor and cancel.

17 $\frac{x^3+y^3}{a^2+2ab+b^2} \cdot \frac{(a+b)^2}{x^2+2xy+y^2}$

18 $\frac{ax+bx}{a^2+2ab+b^2} \cdot \frac{a^2-b^2}{x}$

Division of Fractions

Lesson 5

If $\frac{a}{b}$ is to be divided by $\frac{x}{y}$.

$\frac{a}{b} \div \frac{x}{y} = z$ (where z is the quotient.)

Multiplying the quotient by the divisor gives the dividend.

Or $\frac{a}{b} = z \cdot \frac{x}{y}$.

Multiplying both members by $\frac{y}{x}$.

$$\frac{a}{b} \cdot \frac{y}{x} = z \cdot \frac{x}{y} \cdot \frac{y}{x}$$

Or $\frac{a}{b} \cdot \frac{y}{x} = z$

The conclusion is that *to divide a fraction by a fraction, one may invert the divisor and multiply.*

Ex. To divide $\frac{6a^2}{5b}$ by $\frac{6ab}{2c}$

$$\frac{6a^2}{5b} \div \frac{6ab}{2c} = \frac{6a^2}{5b} \cdot \frac{2c}{6ab} = \frac{12a^2c}{30ab^2}$$

Ex. $$\frac{6x^2+3y}{4x-2} \div \frac{2x-y}{3x^2+y^2} = \frac{6x^2+3y}{4x-2} \cdot \frac{3x^2+y^2}{2x-y} = \frac{(6x^2+3y)(3x^2+y^2)}{(4x-2)(2x-y)}$$

Ex. $\left(2+\frac{3x}{y}\right) \div \left(2x-\frac{6x^2}{y}\right)$

$$\left(2+\frac{3x}{y}\right) \div \left(2x-\frac{6x^2}{y}\right) = \frac{2y+3x}{y} \div \frac{2xy-6x^2}{y}$$

$$= \frac{2y+3x}{y} \cdot \frac{y}{2xy-6x^2} = \frac{2y+3x}{2xy-6x^2}.$$

Problems

1 $\frac{2}{x} \div 2.$

2 $\frac{2a}{4b} \div 3.$

3 $\frac{a-x}{3} \div \frac{1}{2}.$

4 $\frac{2+a}{a-b} \div \frac{a+b}{a-b}.$

5 $\frac{1}{9+c} \div \frac{a+b}{9-c}.$

6 $\frac{a^2-b^2}{a^2+2ab+b^2} \div \frac{a+b}{a-b}$

Note Cancel whenever possible.

7 $\frac{x^2+11x+30}{x^2+10x+25} \div \frac{x+6}{x+5}.$

8 $\frac{2x+4}{3} \div \frac{x+2}{6}.$

9 $\frac{a^3-b^3}{3a^2+6a^4+18a^2} \div \frac{a^2+ab+b^2}{3a^2}$

10 $\dfrac{(a-b)^2}{4x} \div \dfrac{a^2-2ab+b^2}{12x}$.

11 $\dfrac{18a^2}{a+b} \div \dfrac{6a}{a^2-b^2}$.

12 $\dfrac{2}{a-x} \div \dfrac{2a^2-4a^2}{(a-x)^2}$.

Complex Fractions

Lesson 6

1 A complex fraction is one in which fractional terms are involved in one or both its members.

To simplify a complex fraction, simplify each member separately.

Ex. Simplify $\dfrac{a}{1+\dfrac{a}{b}}$.

$$\frac{a}{1+\frac{a}{b}}=\frac{a}{\frac{b+a}{b}}=a\div\frac{b+a}{b}=a.\frac{b}{b+a}=\frac{ab}{b+a}.$$

Ex. Simplify $\dfrac{\dfrac{a}{b}}{1+\dfrac{1}{\frac{a}{b}}}=\dfrac{\frac{a}{b}}{1+\left(1\div\frac{a}{b}\right)}=\dfrac{\frac{a}{b}}{1+\left(1.\frac{b}{a}\right)}=$

$$\frac{\frac{a}{b}}{1+\frac{b}{a}}=\frac{\frac{a}{b}}{\frac{a+b}{a}}=\frac{a}{b}.\frac{a}{a+b}=\frac{a^2}{ab+b^2}.$$

Ex. Simplify $\dfrac{\frac{x}{y}+\frac{c}{d}}{\frac{c}{d}-\frac{x}{2y}}$.

$$\frac{\frac{x}{y}+\frac{c}{d}}{\frac{c}{d}-\frac{x}{2y}}=\frac{\frac{xd+yc}{yd}}{\frac{2cy-dx}{2dy}}=\frac{xd+yc}{yd}\div\frac{2cy-dx}{2yd}=\frac{xd+yc}{yd}.$$

$$\frac{2dy}{2cy-dx}=\frac{2xd+2yc}{2cy-dx}.$$

Ex. Simplify $\dfrac{2-\dfrac{y}{x}}{y+\dfrac{1}{x}}$

$$\frac{2-\frac{y}{x}}{y+\frac{1}{x}}=\frac{\frac{2x-y}{x}}{\frac{xy+1}{x}}=\frac{2x-y}{x}\div\frac{xy+1}{x}=\frac{2x-y}{x}\cdot\frac{x}{xy+1}=\frac{2x-y}{xy+1}.$$

Ex. Simplify $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}$

$$\frac{1}{1+\frac{1}{1+\frac{1}{x}}}=\frac{1}{1+\frac{1}{\frac{x+1}{x}}}=\frac{1}{1+\left(1\div\frac{x+1}{x}\right)}=\frac{1}{1+\left(1\cdot\frac{x}{x+1}\right)}=$$

$$\frac{1}{1+\frac{x}{x+1}}=\frac{1}{\frac{x+1+x}{x+1}}=\frac{1}{\frac{2x+1}{x+1}}=1\div\frac{2x+1}{x+1}=$$

$$1\cdot\frac{x+1}{2x+1}=\frac{x+1}{2x+1}.$$

Problems

Simplify

1. $\dfrac{1}{1+\dfrac{1}{x}}$

2. $\dfrac{x+1}{2x-\dfrac{1}{x+1}}$

3. $\dfrac{2a-4b}{6a+\dfrac{3}{1+a}}$

4. $\dfrac{3-\dfrac{1}{x}}{\dfrac{2}{x}-3}$

5. $\dfrac{\dfrac{2+3x}{1-x}-\dfrac{4-x}{1+x}}{\dfrac{3}{2+x}+\dfrac{3-x}{2}}$

6. $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}$

Fractional Equations

Lesson 7

Solution of equations involving fractions.

To solve an equation involving fractions, (1) clear the equation of fractions, (2) solve by methods before mentioned.

Ex. $\frac{2y}{3}+\frac{5y}{2}=\frac{3y}{6}+8$

L.C.M. of 3, 2, 6 = 6.

Writing all terms as fractions having the common denominator,

$$\frac{4y}{6}+\frac{15y}{6}=\frac{3y}{6}+\frac{48}{6}.$$

Multiplying the equation by (6),

$$4y+15y=3y+48.$$

Collecting terms,

$$16y=48.$$

Or $y=3$.

Ex. $2x+\frac{3x+4}{4}=\frac{6x-2}{8}$

L.C.M. of 4, 8 = 8

Writing all terms as fractions having the denominator (8),

$$\frac{16x}{8}+\frac{2(3x+4)}{8}=\frac{(6x-2)}{8}.$$

Multiplying the equation by (8),

Simplifying, $16x+6x+8=6x-2$

Collecting terms, $16x=-10$

Or, $x=-\frac{5}{8}$

Ex. $\frac{3}{x+1}-\frac{4}{x-1}=\frac{8}{x^2-1}$

L.C.M. of $(x+1)$, $(x-1)$, and $(x^2-1)=(x+1)(x-1)=x^2-1$

Writing all terms in equivalent form having the common denominator.

$$\frac{3(x-1)}{x^2-1}-\frac{4(x+1)}{x^2-1}=\frac{8}{x^2-1}$$

Multiplying the equation by (x^2-1),

$$3(x-1)-4(x+1)=8$$

Expanding and simplifying,

$$3x-3-4x-4=8$$

Collecting terms, $-x=15$

Or $x=-15$

Ex. Solve for (x) when (a, c, b) are considered as known values.

$$\frac{x}{a}+\frac{2x}{cb}=\frac{c}{b}+\frac{x}{a}$$

L.C.M. of a, cb, b and $a=acb$

Writing all terms in equivalent form having the denominator acb,

$$\frac{cbx}{acb}+\frac{2ax}{acb}=\frac{ac^2}{acb}+\frac{cbx}{acb}$$

Multiplying by acb,

$$cbx+2ax=ac^2+cbx$$

Collecting terms,

$$cbx+2ax-cbx=ac^2$$

Factoring, $x(cb+2a-cb)=ac^2$

Dividing by $(cb+2a-cb)$, $x=\frac{ac^2}{cb+2a-cb}=\frac{c^2}{2}$

Problems

Solve,

1 $\frac{3}{x}+\frac{4}{2x}=6$

2 $\frac{2x}{6}-\frac{5x}{2}+\frac{3x}{3}=\frac{4}{3}$

3 $2x+3=\frac{1}{2}$

4 $\frac{x}{3}+2x=6$

5 $\frac{5x+3}{a+1}+\frac{2x-3}{2a-1}$

6 $\frac{1}{2y}+\frac{1}{y}=2+\frac{1}{3}+\frac{1}{3y}$

7 $\frac{a-2}{3}-\frac{a+3}{2}=6$

8 $\frac{x+3}{2a}-\frac{x+2}{3a}=\frac{x+5}{5a}$

9 $\frac{3x+2}{6}+\frac{6x-4}{4}=6$

10 $\frac{(a-b)^2}{3}-\frac{a-b}{2}=\frac{x}{3}$

11 $\frac{x+3}{3a-2}=\frac{x+1}{3a+3}$

12 $\frac{3a+2}{6}=\frac{2a}{4}$

13 $3ax+6a=2x+10a$

14 $\frac{x}{a}+\frac{2x}{b}=\frac{x}{a-b}-\frac{3x}{a}$

15 Two men were in business and made a profit of $3,225 in one year. One man received 20% more than the other. What part of the profit did each receive?

16 A parent offered a boy one dollar if the average of his marks was 80% or over. If the boy gets 90% in mathematics, 85% in history, 78% in rhetoric, and 70% in spelling, what must he get in agriculture to have an average of 80%.

SECTION IV

POWERS AND ROOTS

Lesson 1

General definitions.

1 A *power* is a product obtained by using a number two or more times as a factor.

Ex. The second power of $2 = 2 \cdot 2 = 4$
The third power of $3 = 3 \cdot 3 \cdot 3 = 27$

2 The laws of signs hold as they do in multiplication.

Ex. $+2 \cdot +2 = +4$
$-2 \cdot -2 = +4$

3 The *exponent* is a small number written at one side and above a given number to indicate the number of times the given number is to be used as a factor.

Ex. 3^2, 2 is the exponent.
$3^2 = 9$

Ex. $2^3 = 8$. 3 is the exponent.

Ex. a^2 2 is the exponent.

4 The process of multiplication performed on like factors having equal or unequal exponents is performed by adding exponents.

Ex. $3^2 \cdot 3^4 = 3^6$

Ex. $a^2 \cdot a^3 = a^5$

Ex. $4^a \cdot 4^b = 4^{a+b}$

5 The above law is explained as follows:

$a^2 \cdot a^3 = a \cdot a \cdot a \cdot a \cdot a = (a)$ taken five times as a factor.

This is expressed as a^5

$b^3 \cdot b^4 = b^7 = b \cdot b \cdot b \cdot b \cdot b \cdot b \cdot b = (b)$ taken seven times as a factor, and is expressed as b^7.

6 In case the terms have more than one factor, the exponents of like factors are added.

Ex. $a^3x^2 \cdot a^2x^5 = a^5x^7$

$b^2y^3 \cdot by = b^3y^4$

$x^2y^2z^4 \cdot x^3y^5z^6 = x^5y^7z^{10}$

7 Should there be coefficients to one or both terms, these become factors of the product.

Ex. $3a^2b \cdot a^3b = 3a^5b^2$

Ex. $4a^3b^2 \cdot 5ab^3 = 20a^4b^5$

8 The *root* of a number is one of a certain number of equal factors of a number.

Ex. 2 is a root of 4.

Ex. 3 is a root of 27

9 The square root is one of two equal factors.

Ex. 2 is the square root of 4.

The cube root is one of three equal factors.

Ex. 4 is the cube root of 64.

The fourth root is one of four equal factors.

Ex. 2 is the fourth root of 16.

10 The sign $\sqrt{}$ is called the radical sign, and indicates that a root is to be found.

11 A number placed in the $\sqrt{}$ indicates the root.

Ex. $\sqrt[2]{}$ indicates square root.

Ex. $\sqrt[3]{}$ indicates cube root.

Ex. $\sqrt[4]{}$ indicates fourth root.

This number is called the index number.

(*Note*) The square root is usually expressed by writing the sign with no number in the $\sqrt{}$.

Ex. $\sqrt{a}$, $\sqrt{4}$, $\sqrt{16}$

Problems

1 Define power, exponent, radical.

2 Multiply $3x^2y$ by $2y$

3 Multiply $4a^2b^2$ by $2ab$

4 Multiply $10ab^3$ by b^3

5 Multiply $6m^2n$ by mn

6 Multiply m^3n^2 by $5mn^2$

7 Multiply $6x^2y$ by $3xy^2$

8 Express the square root of $3a+b$

9 Express the cube root of $4x-2y$

10 Express the fourth root of $3xy-4$

11 Multiply $16x$ by $3xy$

12 Multiply $4m^2$ by $2n^2$

POWERS

Lesson 2

Powers of monomials

Integral exponents.

1 If a number having an exponent is to be raised to a power, the exponent of the number is multiplied by the number expressing the power.

Ex. $(a^2)^3 = a^6$

Ex. $(a^3)^4 = a^{12}$

Ex. $(a^x)^2 = a^{2x}$

Ex. $(a^x)^y = a^{xy}$

2 The explanation of the above law is as follows:

$a^2 = a \cdot a$

$(a \cdot a)^3 = a \cdot a \cdot a \cdot a \cdot a \cdot a = a^6$

3 If the product of two numbers, each of which has an exponent, is to be raised to a power, the exponent of each factor is multiplied by the number indicating the power.

Ex. $(a^2b^3)^2 = a^4b^6$

Ex. $(x^3y^4)^3 = x^9y^{12}$

Ex. $(4a^2bc^3)^4 = 4^4a^8b^4c^{12}$

4 The above is evident when one considers that a^2b^3 is equivalent to $a \cdot a \cdot b \cdot b \cdot b$. To take this value twice as a factor is to take (a) four times as a factor and (b) six times as a factor.

5 The process of raising an expression to any power is called involution.

6 Involution has to do with the raising of monomial, binomial, and polynomial expressions to powers.

7 The raising of any monomial to a power is accomplished by raising each factor of the monomial to the required power.

Ex. a^2 raised to the second power is $(a^2)^2 = a^4$

Ex. $2a$ raised to the second power is $(2)^2 . (a)^2 = 4a^2$

Ex. $(3a^3b)^2 = 9a^6b^2$

Ex. $(2xy^2)^3 = 8x^3y^6$

Ex. $(4a^{-2}x^n)^2 = 16a^{-4}x^{2n}$

(*Note*) In case there is no literal factor, the number is not usually factored.

Ex. $4^2 = 16$

Ex. $5^3 = 125$

8 The same method is followed, regardless of the seeming difficulty of the problem.

Ex. $(x^2y^{\frac{1}{2}})^3 = x^6y^{\frac{3}{2}}$

Ex. $(4a^0b^{-3})^2 = 16b^{-6}$ (*Note*) $a^0 = 1$

9 To raise a fraction to a power, raise both members of the fraction to the power.

Ex. $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$ Proof: $\left(\frac{2}{3}\right)^2 = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$

Ex. $\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$

Ex. $\left(\frac{2x^2}{3y}\right)^2 = \frac{4x^4}{9y^2}$

10 Monomials having fractional factors may be expanded by using the above rule.

Ex. $\left(\frac{1}{2}a^2bc^3\right)^2 = \frac{1}{4}a^4b^2c^6$

Ex. $\left(\frac{2}{3}a^6bc\right)^3 = \frac{8}{27}a^{18}b^3c^3$

11 In case a monomial is to be raised to a power expressed by a literal exponent, the same rules hold.

Ex. $(a^2)^n = a^{2n}$

Ex. $(aby)^x = a^xb^xy^x$

Ex. $(2a^2b)^y = 2^ya^{2y}b^y$

Problems

1 Define involution.

Expand,

2	$(2x^2)^2$	14	$(ax)^2$
3	$(4xy)^2$	15	$(2ab)^3$
4	$(3xyz)^2$	16	$(2a^n)^n$
5	$(4x^2y)^2$	17	$(x^2y)^{3a}$
6	$(2xy^2)^3$	18	$(\frac{2}{5}a^2b)^2$
7	$(5xy^2z^3)^4$	19	$(\frac{1}{3}c^{\frac{1}{2}})^3$
8	$(8ab^2c)^3$	20	$(2a^2b^{\frac{1}{2}}c^{-2})^2$
9	$(10c^2d)^5$	21	$(4m^2n^{\frac{1}{3}}p^{-2})^3$
10	$(2abc^3)^2$	22	$(a^{\frac{1}{2}}b^2c^{-3})^2$
11	$(3a^{\frac{1}{2}})^2$	23	$(x^ay^bw^c)^{\frac{1}{2}}$
12	$(2x^{\frac{1}{2}}y)^{\frac{1}{2}}$	24	$(2m^2n)^{\frac{1}{2}}$
13	$(\frac{2}{3})^2$	25	$(4a^{\frac{1}{2}}b^{-2}c^c)^{\frac{1}{2}}$

Lesson 3

Fractional exponents.

1 If $a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{3}{3}} = a^{2}$

Then a^{2} is the product of three equal factors, and from the definition of a root,

$a^{\frac{1}{3}}$ is the cube root of a^{2}

Or $\sqrt[3]{a^{2}} = a^{\frac{2}{3}}$

Stated in general terms, the numerator of a fractional exponent indicates the power to which a number is to be raised, and the denominator indicates the root to be taken.

Ex. $a^{\frac{1}{4}}$ means that the 4th root of a^{3} is to be taken.

Ex, $x^{\frac{3}{3}}$ means that the 3rd root of x^{2} is to be taken.

2 In the case of a product of two or more factors, each having a fractional exponent, the root can be expressed as follows:

Ex. $a^{\frac{1}{3}} \cdot b^{\frac{1}{4}} = a^{\frac{4}{12}} b^{\frac{3}{12}} = \sqrt[12]{a^{4} b^{3}}$

Ex. $a^{\frac{2}{3}} \cdot b^{\frac{1}{4}} = a^{\frac{8}{12}} x^{\frac{3}{12}} = \sqrt[12]{a^{8} b^{3}}$

Ex. $x^{\frac{1}{5}} \cdot y^{\frac{2}{4}} = x^{\frac{4}{20}} \cdot y^{\frac{10}{20}} = \sqrt[20]{x^{4} y^{10}}$

In case an indicated root is raised to a power.

Ex. $(\sqrt[3]{x})^{4} = (x^{\frac{1}{3}})^{4} = x^{\frac{4}{3}} = \sqrt[3]{x^{4}}$

Ex. $(\sqrt[2]{x})^{5} = (y^{\frac{1}{2}})^{5} = y^{\frac{5}{2}} = \sqrt{y^{5}}$

Ex. $(\sqrt[4]{a})^{3} = (a^{\frac{1}{4}})^{3} = a^{\frac{3}{4}} = \sqrt[4]{a^{3}}$

3 In case a root is to be taken of a power.

Ex. $\sqrt{a^{\frac{1}{2}}} = a^{\frac{1}{4}} = \sqrt[4]{a}$

Ex. $\sqrt[3]{a^2} = a^{\frac{2}{3}} = \sqrt[9]{a^2}$

Ex $\sqrt{b^{\frac{1}{2}}} = b^{\frac{1}{4}} = \sqrt[8]{b}$

(The index number becomes a factor of the denominator.)

Problems

Write with fractional exponents.

1	$\sqrt{a}$	7	$\sqrt{2a}$
2	$\sqrt{x^2}$	8	$\sqrt{m^3}$
3	$\sqrt{2^3}$	9	$\sqrt[3]{x^2}$
4	$\sqrt[3]{a}$	10	$\sqrt[3]{m^4}$
5	$\sqrt[4]{x^3}$	11	$\sqrt[a]{x^b}$
6	$\sqrt[5]{2a^4}$	12	$\sqrt[a]{a^y}$
		13	$\sqrt[3]{a^{\frac{1}{2}}}$

Write with the radical sign.

14	$a^{\frac{1}{2}}$	18	$a^{\frac{1}{2}}b^{\frac{1}{2}}$
15	$b^{\frac{2}{3}}$	19	$2^{\frac{2}{3}}$
16	$c^{\frac{1}{5}}$	20	$a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{2}}$
17	$(ab)^{\frac{1}{2}}$	21	$(abc)^{\frac{a}{b}}$
		22	$(xy)^{\frac{m}{n}}$

Lesson 4

Zero and negative exponents.

1 Not only do we have integral and fractional exponents to work with but also the zero exponents often appear.

Any number taken zero times as a factor gives the product one.

Ex. $2^0=1$

Ex. $a^0=1$

Ex. $(2abc)^0=1$

2 The explanation of this law is as follows:

From the first law of exponents $a^x . a^0 = a^{x+0} = a^x$

Dividing the equation through by a^x, $a^0 = \frac{a^x}{a^x} = 1$

Another explanation is this,—as the unit (one) is always used as the base for multiplication, to use a number zero times as a factor means that the original (one) is not multiplied by anything. Therefore the result is simply one.

Ex. $3 \cdot 4$ = four units taken as many times as one unit is taken to get three.

$3^2 = 3 \cdot 3$, indicates that 3 units are taken twice as a factor.

3 Any number having a negative exponent is equal to one divided by the number with a positive exponent.

Ex. $a^{-2} = \frac{1}{a^2}$

Ex. $x^{-3} = \frac{1}{x^3}$

4 The proof of this law follows:

$a^{-m} \cdot a^{+m} = a^{-m+m} = a^0 = 1$

$a^{-m} \cdot a^m = 1$

Dividing through by a^m $\quad a^{-m} = \dfrac{1}{a^m}$

5 This law is made use of in rewriting an expression having a fractional form in an integral form.

Ex. $\dfrac{a^2}{x^3} = a^2 x^{-3}$

Ex. $\dfrac{x^3 y^4}{w^{-4} z^2} = x^3 y^4 w^4 z^{-2}$

(*Note*) Integral expressions may also be written in fractional form by reversing the process.

Problems

Expand

1 $(2a^0b)^2$

2 $(3ab^0)^0$

3 $(a^0)^2$

4 $(x^0b)^3$

5 $(2a^0b^0)^2$

6 $(2ab)^0$

7 $(xyz)^0$

8 $(x^2yz)^2$

9 $3a^0 = ?$

10 $4a^0b$

Write with positive exponents.

11 a^{-2}

12 $(ab)^{-3}$

13 a^2b^{-3}

14 a^0b^{-4}

15 $a^2b^3c^{-2}$

16 $x^{-3}y^{-4}$

17 $m^{-2}n^{-3}$

18 $a^{-x}b^{-y}$

19 $x^{-4}y^{-2}$

20 $x^2y^{-2}z^{-3}$

Lesson 5

Powers of binomials

Special rules.

1 It often happens that one wishes to raise a binomial to some power, usually to the second or third, but often to the fourth, fifth, or even higher power.

A formula may be developed by which all such powers may be found, but for the present purposes the forms of the second and third powers will be memorized.

As stated once before, under the special rules for multiplication, the rule for raising any binomial to the second power is,

Square the first term.
Add twice the product of the two terms.
Add the square of the second term.

(*Note*) This law is based upon the fact that every multiplication of a binomial by itself gives such an expression.

Ex. $(a+b)^2=a^2$ Square of first term.
$+2ab$ Twice the product of two terms.
$+b^2$ Square of last term.
Or, $(a+b)^2=a^2+2ab+b^2$

Ex. $(2a-b)^2=4a^2$ Square of first term.
$+2(2a)\ (-b)$ Twice product of two terms.
$+b^2$ Square of last term.
Or $(2a-b)^2=4a^2-4ab+b^2$

(*Note*) It will be noticed that twice the product of the two terms gives a minus quantity.

Ex. $(x-4y)^2=x^2$ Square of first term.
$+2(x)(-4y)$ Twice the product of two terms.
$+16y^2$ Square of the second term.
Or $(x-4y)^2=x^2-8xy+16y^2$

Ex. $(-x-y)^2=x^2$ Square of first term.
$+2(-x)(-y)$ Twice the product of two terms.
$+y^2$ Square of second term.
Or $(-x-y)^2=x^2+2xy+y^2$

2 The rule for raising a binomial to the third power is,

The cube of the first term.

Plus three times the product of the first term squared and the 2nd term.

Plus three times the product of the first term and the 2nd term squared.

Plus the cube of the second term.

Ex. $(a+b)^3=a^3$ The cube of the first term.
$+3a^2b$ Three times the product of the 1st term squared by the second term.
$+3ab^2$ Three times the product of the 1st term by the second term squared.
$+b^3$ The cube of the second term.
Or $(a+b)^3=a^3+3a^2b+3ab^2+b^3$

(*Note*) This rule is derived inductively from the fact that actual multiplication in a great number of problems gives the same results.

Ex. $(2a-b)^3=8a^3$ The cube of the 1st term.
$+3(2a)^2(-b)$ Three times the product of the 1st term squared by the 2nd term.

$+3(2a)(-b)^2$ Three times the product of the 1st term by the 2nd term squared.

$+(-b)^3$ The cube of the 2nd term.

Or $(2a-b)^3=8a^3-12a^2b+6ab^2-b^3$

Ex. $(-6-2x)^3=(-6)^3$ The cube of the 1st term.

$+3(-6)^2(-2x)$ Three times the product of the 1st term squared by the 2nd term.

$+3(-6)(-2x)^2$ Three times the product of the 1st term by the 2nd term squared.

$+(-2x)^3$ The cube of the 2nd term.

Or $(-6-2x)=-216-216x-72x^2-8x^3$

3 It is found by multiplication that the fourth power of a binomial is equal to

(1) the fourth power of the first term.

(2) four times the cube of the first term multiplied by the second term.

(3) six times the square of the first term multiplied by the square of the second term.

(4) four times the first term multiplied by the cube of the second term.

(5) the fourth power of the second term.

Ex. $(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$

Ex. $(x+y)^4=x^4+4x^3y+6x^2y^2+4xy^3+y^4$

Ex. $(m+n)^4=m^4+4m^3n+6m^2n^2+4mn^3+n^4$

4 To overcome the difficulty of expanding the difference of two numbers to any power, it is best to first write a general form then make substitutions.

Ex. $[()+()]^2=()^2+2()()+()^2$

Ex. $[()+()]^3=()^3+3()^2()+3()()^2+()^3$

Ex. $[()+()]^4=()^4+4()^3()+6()^2()^2+4()()^3+()^4$

If these forms are studied, it will be seen that the exponents of the first term decrease regularly by one, and the exponents of the second term increase regularly by one. The coefficients may be memorized.

Suppose $(x-y)^2$ is to be expanded.

Expanding each term and combining,

$(x)^2+2(x)(-y)+(-y)^2$

$x^2-2xy+y^2$

If $(a-b)^2$ is to be expanded,

$(a)^2+2(a)(-b)+(-b)^2$

Expanding each term and combining,

$a^2-2ab+b^2$

Problems

Expand

1 $(a+b)^2$
2 $(a-b)^2$
3 $(x-y)^2$
4 $(2x+1)^2$
5 $(1-3a)^2$
6 $(2x+y)^2$
7 $(2a+3b^2)^2$
8 $(1-3ax^2)^2$
9 $(a-b)^3$
10 $(a+b)^3$
11 $(1-b)^3$
12 $(x+1)^3$
13 $(2x-1)^3$
14 $(1-2x)^3$
15 $(a-2b)^3$
16 $(m+n)^3$
17 $(a+b)^4$
18 $(a-b)^4$
19 $(x+y)^4$
20 $(x-y)^4$
21 $(m+n)^4$
22 $(x-1)^4$
23 $(s+1)^4$
24 $(2x-b)^4$
25 $(1+x)^4$

Lesson 6

General rule.

Any binomial may be raised to any power by application of a general rule developed inductively from the special cases. This rule is proved and discussed in a later discussion. It is here given in working form only.

Given $(a+b)^n$

(*Note*) (n) is used as the exponent to stand for a general number.

1 Write a with exponent (n) or a^n

2 The coefficient of the second term is n.

The (a) in the second term has the exponent $(n-1)$.

The (b) in the second term has the exponent (1)

The expansion thus far is $a^n+na^{n-1}b+$ ————

3 The third term has a coefficient formed by multiplying the coefficient of the second term by the exponent of (a) in the second term, and dividing this product by 2. (a) takes the exponent $(n-2)$, (b) takes the exponent (2).

The expansion thus far is $a^n+na^{n-1}b+\frac{n(n-1)}{2}$ $a^{n-2}b^2+$ ————

4 The fourth term has a coefficient formed by multiplying the coefficient of the third term by the exponent of a and dividing by (3). (a) takes the exponent $(n-3)$, (b) takes the exponent (3)

The expansion to this point is,

$$a^n+na^{n-1}b+\frac{n(n-1)}{2}a^{n-2}b^2+\frac{n(n-1)\ (n-2)}{2.3}a^{n-3}b^3$$

5 The exponents for each term decrease regularly with respect to (a), and increase regularly with respect to (b)

The coefficient of a next term is found from the last preceding term by multiplying the coefficient by the exponent of the first term of the binomial, and dividing by the number of the term.

Ex. $(a+b)^{10}$

First term, a^{10}

Second term, $10a^9b$

Third term, $\frac{10.9}{2}a^8b^2$

Fourth term, $\frac{10.9.8}{2.3}a^7b^3$

etc.

6 In case the binomial is made up of terms of more than one factor or terms of different signs, it is advisable to use a parentheses form in the expansion.

Ex. $(2a-b^2)^6$

First developing the expansion of $(\ )+(\ ))^6$, we have,

$$(\ )^6+6(\ )^5(\ )+15(\ )^4(\ )^2+20(\ )^3(\ )^3+15(\ )^2(\ )^4+6(\ )(\ )^5+(\ )^6$$

Filling the parentheses,

$$(2a)^6+6(2a)^5(-b^2)+15(2a)^4(-b^2)^2+20(2a)^3(-b^2)^3+15(2a)^2(-b^2)^4+6(2a)(-b^2)^5+(-b^2)^6$$

Expanding,

$64a^6-192a^5b^2+240a^4b^4-160a^3b^6+60a^2b^8-12ab^{10}+b^{12}$

Ex. $(3a^2-b)^3$

$()^3+3()^2()+3()()^2+()^3$

$(3a^2)^3+3(3a^2)^2(-b)+3(3a^2)(-b)^2+(-b)^3$

Expanding,

$27a^6-27a^4b+9a^2b^2-b^3$

Any polynomial may be squared by squaring each term, and taking twice the product of each term by every other term. The signs of the terms determine the signs of the expansion.

Ex. $(a+b+c)^2$

$a^2+b^2+c^2+2ab+2ac+2bc$

Ex. $(a-b+c)^2$

$a^2+b^2+c^2-2ab+2ac-2bc$

This is evident from the fact that the multiplication of an expression by itself allows two, and only two, like products of terms.

Ex.
$$\begin{array}{l} a+b+c \\ a+b+c \\ \hline a^2+ab+ac \\ +b^2\quad +ab\quad +bc \\ \qquad\quad +ac+bc+c^2 \\ \hline b^2+a^2+2ab+2ac+2bc+c^2 \end{array}$$

Problems

Expand

1 $(a+b)^2$

2 $(a+b)^3$

3 $(a+b)^4$

4 $(a+b)^5$

5 $(x-y)^3$
6 $(a-b)^4$
7 $(m+n)^3$
8 $(1+x)^2$
9 $(2x+3)^2$
10 $(1+3y)^5$
11 $(2x+3y)^6$
12 $(a-2)^4$
13 $(-x-y)^2$
14 $(3-2y)^2$
15 $(2ax+2y)^3$
16 $(3abc-2d)^2$
17 $(1+2x)^3$
18 $(3-2b)^3$

ROOTS

Lesson 1

Monomials

1 Rational roots.

The radical sign ($\sqrt{\ }$) placed over an expression indicates that a root is to be taken. The number placed in the ($\sqrt{\ }$) indicates the exact root to be taken. This number is called the index number.

Ex. $\sqrt[3]{a^3b^4}$ indicates that the third root of a^3b^4 is to be taken.

An indicated root that can be exactly found is called a *rational* root.

Ex. $\sqrt[3]{8}=2$

Ex. $\sqrt[2]{4}=2$

Ex. $\sqrt[3]{27}=3$

Ex. $\sqrt[4]{a^4}=a$

(*Note*) If no index number is given the root to be found is the second.

2 Irrational roots.

An indicated root that cannot be exactly found is called an *irrational* root.

Ex. $\sqrt[3]{16}$, $\sqrt{a^3}$, $\sqrt[4]{32}$, etc.

Indicated roots that cannot be exactly found can only be approximated.

An expression involving an irrational root may be simplified by one of three methods.

(*a*) When the exponent of the quantity beneath the radical sign is a factor of the index number, the index number may be divided by the exponent.

Ex. $\sqrt[4]{a^2} = \sqrt[2]{a}$

Ex. $\sqrt[6]{a^3} = \sqrt[2]{a}$

Ex. $\sqrt[4]{4} = \sqrt[4]{2^2} = \sqrt[2]{2}$

(*Note*) Often the quantity beneath the radical sign may be written in an equivalent form with an exponent.

Ex. $\sqrt[6]{8} = \sqrt[6]{2^3} = \sqrt[2]{2}$

Problems

Simplify

1 $\sqrt{16}$

2 $\sqrt{25}$

3 $\sqrt{a^2b^4}$

4 $\sqrt{a^6}$

5 $\sqrt[4]{4}$

6 $\sqrt[6]{9}$

7 $\sqrt[4]{25}$

8 $\sqrt[8]{49}$

9 $\sqrt[10]{16}$

10 $\sqrt[4]{a^2}$

11 $\sqrt[6]{b^3}$

12 $\sqrt[8]{a^4b^4}$

13 $\sqrt[4]{(a-b)^2}$

14 $\sqrt[6]{4a^2}$

15 $\sqrt[12]{81}$

16 $\sqrt[10]{32}$

Lesson 2

Irrational Roots—Continued

(*b*) If the quantity beneath the radical sign is factorable, one factor may be a perfect power. In such case its root is written outside the radical sign.

Ex. $\sqrt{18} = \sqrt{9 \cdot 2} = \sqrt{9} \cdot \sqrt{2} = 3\sqrt{2}$

Ex. $\sqrt[3]{54} = \sqrt[3]{27 \cdot 2} = \sqrt[3]{27} \cdot \sqrt[3]{2} = 3\sqrt[3]{2}$

Ex. $\sqrt{2a^2} = \sqrt{2} \cdot \sqrt{a^2} = \sqrt{2} \cdot a = a\sqrt{2}$

Ex. $\sqrt[3]{a^4} = \sqrt[3]{a^3} \cdot \sqrt[3]{a} = a\sqrt[3]{a}$

Problems

1	$\sqrt{8}$	9	$\sqrt{8a^3}$
2	$\sqrt{18}$	10	$\sqrt{a^3b^5}$
3	$\sqrt{40}$	11	$\sqrt{50}$
4	$\sqrt{12}$	12	$\sqrt[3]{54}$
5	$\sqrt{20}$	13	$\sqrt[3]{16}$
6	$\sqrt{98}$	14	$\sqrt[4]{32}$
7	$\sqrt{a^3}$	15	$\sqrt[3]{x^4}$
8	$\sqrt{a^5}$	16	$\sqrt[4]{(a-b)^5}$

Lesson 3

Irrational roots

(c) If the quantity beneath the radical sign is a fraction, the expression may be simplified by multiplying both members of the fraction by such a number as will make the denominator a perfect power.

Ex. $\sqrt{\frac{1}{2}} = \sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2} = \frac{1}{2}\sqrt{2}.$

Ex. $\sqrt{\frac{1}{3}} = \sqrt{\frac{3}{9}} = \frac{\sqrt{3}}{\sqrt{9}} = \frac{\sqrt{3}}{3} = \frac{1}{3}\sqrt{3}.$

Ex. $\sqrt[3]{\frac{1}{a}} = \sqrt[3]{\frac{a^2}{a^3}} = \frac{\sqrt[3]{a^2}}{\sqrt[3]{a^3}} = \frac{\sqrt[3]{a^2}}{a} = \frac{1}{a}\sqrt[3]{a^2}.$

Ex. $\sqrt{\frac{2}{3}} = \sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{\sqrt{9}} = \frac{\sqrt{6}}{3} = \frac{1}{3}\sqrt{6}.$

Problems

Simplify

1 $\sqrt{\frac{1}{2}}$

2 $\sqrt{\frac{3}{4}}$

3 $\sqrt{\frac{2}{3}}$

4 $\sqrt{\frac{a}{b}}$

5 $\sqrt{\frac{1}{5}}$

6 $\sqrt{\frac{x}{y}}$

7 $\sqrt{\frac{1}{7}}$

8 $\sqrt{\frac{1}{2}}$

9 $\sqrt{\frac{8}{5}}$

10 $\sqrt{\frac{ab}{c}}$

11 $\sqrt{\frac{2ab}{b^2}}$

12 $\sqrt{\frac{3ab^2}{b^3}}$

Lesson 4

Reduction to same order

(d) Radical expressions may be reduced to the same order (having the same index numbers), by

1 Writing the expressions with fractional exponents.

2 Writing the exponents as equivalent fractions having common denominators.

3 Writing the expressions with the radical signs.

Ex. $\sqrt{a}$, $\sqrt[3]{a^2}$, $\sqrt[4]{a^3}$

Writing with fractional exponents, $a^{\frac{1}{2}}$, $a^{\frac{2}{3}}$, $a^{\frac{3}{4}}$

Writing the exponents as equivalent exponents having a common denominator.

$a^{\frac{6}{12}}$, $a^{\frac{8}{12}}$, $a^{\frac{9}{12}}$

Writing the expressions in radical form.

$\sqrt[12]{a^6}$, $\sqrt[12]{a^8}$, $\sqrt[12]{a^9}$

The expressions now have common index numbers.

Problems

Reduce

1 $\sqrt{2}$, $\sqrt[3]{3}$

2 $\sqrt{a}$, $\sqrt[4]{b}$

3 $\sqrt[3]{2}$, $\sqrt[3]{3}$

4 $\sqrt{2}$, $\sqrt[3]{2}$, $\sqrt[4]{2}$

5 $\sqrt{a}$, $\sqrt[3]{b}$

6 $\sqrt[3]{m}$, $\sqrt{m}$, $\sqrt[4]{m}$

7 $\sqrt{6}$, $\sqrt[3]{6}$,

8 $\sqrt{2}$, $\sqrt[3]{3}$

Lesson 5

Mixed numbers to entire surds

(e) A mixed expression may be written as an entire surd by raising the rational factor to the power indicated by the index number, and placing it beneath the radical sign as a factor.

Ex. $2\sqrt{3} = \sqrt{4 \cdot 3} = \sqrt{12}$

Ex. $6\sqrt{a} = \sqrt{36 \cdot a}$

Ex. $3\sqrt{b^3} = \sqrt{9 \cdot b^3}$

Problems

Reduce

1	$3\sqrt{2}$	9	$3b\sqrt{ab}$
2	$2\sqrt{5}$	10	$4xy\sqrt{2a}$
3	$5\sqrt{6}$	11	$3a\sqrt{b}$
4	$a\sqrt{b}$	12	$2x\sqrt{yw}$
5	$3\sqrt{10}$	13	$6x^2\sqrt{xy}$
6	$a\sqrt{abc}$	14	$3\sqrt{2}$
7	$x\sqrt{y}$	15	$2\sqrt{2}$
8	$2x\sqrt{w}$	16	$8\sqrt{6}$

Lesson 6

Application of the fundamental operations to radical expressions.

(1) Radical expressions may be added or subtracted, provided they have like radicands and like index numbers.

Ex. $2\sqrt{a}+3\sqrt{a}=5\sqrt{a}$

Ex. $6\sqrt[3]{3b}+4\sqrt[3]{3b}=10\sqrt[3]{3b}$

Ex. $3\sqrt{b}-2\sqrt{b}+4\sqrt{b}=5\sqrt{b}$

If expressions have not like radical factors, it is often possible to reduce them to forms having like radical factors.

Ex. $\sqrt{2}+6\sqrt{18}-3\sqrt{8}$

Reducing, $2\sqrt{2}+6\cdot 3\sqrt{2}-3\cdot 2\sqrt{2}$

Or $2\sqrt{2}+18\sqrt{2}-6\sqrt{2}=14\sqrt{2}$

Ex. $8\sqrt{27}-2\sqrt{12}+4\sqrt{48}$

Reducing, $8\cdot 3\sqrt{3}-2\cdot 2\sqrt{3}+4\cdot 4\sqrt{3}$

Or, $24\sqrt{3}-4\sqrt{3}+16\sqrt{3}$

Combining terms, $36\sqrt{3}$

(2) Radical expressions may be multiplied or divided, provided the index numbers are like.

Ex. $\sqrt{3}\cdot\sqrt{5}=\sqrt{15}$

Ex. $\sqrt{a}\cdot\sqrt{b}\cdot\sqrt{c}=\sqrt{a\cdot b\cdot c\cdot}$

Ex. $2\sqrt[3]{3}\cdot 4\sqrt[3]{a}\cdot 6\sqrt[3]{3b}=48\sqrt[3]{9ab}$

Radical expressions may be reduced to equivalent expressions with like index numbers by the method explained in the previous paragraph.

Ex. $\sqrt{a}.\sqrt[3]{b}.\sqrt[4]{c}$

Writing with fractional exponents $a^{\frac{6}{12}}.b^{\frac{4}{12}}.c^{\frac{3}{12}}$

Reducing the exponents to equivalent fractions having a common demoninator.

Writing in radical form,

$\sqrt[12]{a^6}.\sqrt[12]{b^4}.\sqrt[12]{c^3}$

Multiplying, $\sqrt[12]{a^6 b^4 c^3}$

Problems

Combine

1 $\sqrt{18}+\sqrt{8}+\sqrt{15}$

2 $\sqrt{27}+\sqrt{48}-\sqrt{108}$

3 $\sqrt{20}-\sqrt{180}+\sqrt{320}$

4 $\sqrt{a^2 b}+\sqrt{c^2 b}+\sqrt{d^2 b}$

5 $\sqrt{63}-\sqrt{28}+\sqrt{252}$

6 $\sqrt[3]{16}+\sqrt[3]{54}-\sqrt[3]{128}$

7 $-\sqrt{54}-\sqrt{24}+\sqrt{150}+\sqrt{96}$

8 $\sqrt[4]{16a}+\sqrt[4]{81a}-\sqrt[4]{256a}$

9 $\sqrt{36x^3}-\sqrt{9x^3}-\sqrt{4x^3}$

10 $\sqrt{a^2 xy}+\sqrt{4a^2 xy}-\sqrt{16a^2 xy}$

Simplify

1 Multiply $\sqrt{3}\cdot\sqrt{2}\cdot\sqrt{4}$

2 " $\sqrt[3]{5}\cdot\sqrt{2}$

3 " $\sqrt{7}\cdot\sqrt[3]{2}\cdot\sqrt{3}$

4 " $\sqrt[4]{2}\cdot\sqrt[3]{3}\cdot\sqrt{4}$

5 Multiply $\sqrt[3]{4}\cdot\sqrt[3]{3}\cdot\sqrt{2}$

6 Divide $\sqrt{3}$ by $\sqrt{2}$

7 " $\sqrt{3}$ by $\sqrt[3]{3}$

8 " $\sqrt[3]{9}$ by $\sqrt{2}$

9 " $\sqrt{5}$ by $\sqrt[3]{4}$

10 " $\sqrt{28}$ by $\sqrt{18}$

11 Multiply $\sqrt{x}\cdot\sqrt[3]{xy}$

12 Divide $\sqrt{2x^3y}$ by $\sqrt[3]{xy^2}$

Lesson 7

Polynomials

The square root of a polynomial.

By applying the rule for squaring a polynomial inversely, it is possible to test any expression for a square root as follows:

Ex. $\sqrt{a^4+4a^3+4a+1+6a^2}$

Arranging the terms in descending order of exponents,

$$a^4+4a^3+6a^2+4a+1$$

Extracting the square root of the first term for the first term of the quotient,

$$a^4+4a^3+6a^2+4a+1 \,|\underline{a^2}$$

Squaring the quotient and subtracting this result from the expression,

$$a^4+4a^3+6a^2+4a+1 \,|\underline{a^2}$$
$$\underline{a^4}$$
$$4a^3+6a^2+4a+1$$

Dividing the first term of the remainder by twice the first term of the quotient,

$$a^4+4a^3+6a^2+4a+1 \,|\underline{a^2+2a}$$
$$\underline{a^4}$$
$$\underline{2a^2}\,|\,4a^3+6a^2+4a+1$$

Bringing this term $(2a)$ down as the second term of the trial divisor, and multiplying by $(2a)$.

$$\begin{array}{r|l} & a^4+4a^3+6a^2+4a+1 \;\underline{|a^2+2a+1} \\ & \underline{a^4} \\ \underline{2a^2+2a} & 4a^3+6a^2+4a+1 \\ & \underline{4a^3+4a^2} \\ \underline{2a^2+4a+1} & 2a^2+4a+1 \\ & 2a^2+4a+1 \end{array}$$

(*Note*) After the remainder $2a^2+4a+1$ has been found, the previous process is repeated.

A The quotient (a^2+2a) is multiplied by 2

B $(2a^2)$ is contained once in $(2a^2)$. This one is placed in the quotient and added to the trial divisor.

C The trial divisor is then multiplied by the (one), and subtracted from $(2a^2+4a+1)$, the previous remainder.

Ex. To extract the square root of $4x^4+4x^3-3x^2-2x+1$

$$\begin{array}{r|l} & 4x^4+4x^3-3x^2-2x+1 \;\underline{|2x^2+x-1} \\ & \underline{4x^4} \\ \underline{4x^2+x} & 4x^3-3x^2-2x+1 \\ & \underline{4x^3+x^2} \\ \underline{4x^2+2x-1} & -4x^2-2x+1 \\ & \underline{-4x^2-2x+1} \end{array}$$

Problems

Test for the square root.

1 $x^4-6x^3+11x^2-6x+1$

2 $4x^4+4x^3+5x^2+2x+1$

3 $a^6-2a^5+3a^4-4a^3+3a^2-2a+1$

4 $4a^2-4ab+4a+b^2-2b+1$

5 $1+2x+3x^2+2x^3+x^4$

Lesson 8

Arithmetical numbers

The square root of numbers may be taken as follows:

Ex. 144

Point off from the right in groups of two digits each.

1'44

Determine the nearest square root to the number in the first group. It is 1.

144 | 1
1
—
— | 44

Place the square of the quotient under the first group and subtract. The remainder is 44.

Multiply the quotient by two for the first part of the trial divisor.

1'44 | 1
1
—
2 | 44

2 is contained in 4 twice. Place this number in the quotient and also in the trial divisor, then multiply the trial divisor by it and subtract.

1'44 | 12
1
—
22 | 44
44
—

Ex Take the square root of 625

Point off 6'25

Determining the nearest square, 6'25 | 2
4
225

Multiply the quotient by 2 for the first trial divisor.

6'25 | 2
4
4 | 225

4 is contained 5 times in 22. Place 5 in the quotient and in the trial divisor, then multiply by 5 and subtract.

6'25 | 25
4
45 | 225
225

Ex. $\sqrt{25281}$

Point off 2'52'81

Determine the nearest square in the first group.

2'52'81 | 1
1
152

Multiply (1) by (2) for the first trial divisor.

2'52'81 | 1
1
2 | 152

Although (2) is contained (7) times in (15), it is not possible to use (7), owing to the fact that the adding of (7) to the trial divisor will give too large a

number when multiplied by (7). (5) is the largest number we can use.

```
      2'52'81 |15
      1
    ----
 25 | 152
      125
     -----
       2781
```

Multiplying the quotient by (2) for the new trial divisor, the process is finished as follows:

```
      2'52'81 |159
      1
    ----
 25 | 152
      152
     -----
309 | 2781
      2781
     -----
```

(*Note*) 30 is contained in 278 nine times. This nine is placed in the quotient and in the trial divisor. The trial divisor is then multiplied by 9, giving 2781.

Problems

1	196	7	2304
2	484	8	3645
3	466489	9	18496
4	8364	10	231
5	68459	11	1002001
6	1024	12	63945

Lesson 9

The Imaginary number

(1) If $(+a)$ and $(+a)$ are multiplied together the product is $(+a^2)$

If $(-a)$ and $(-a)$ are multiplied together the product is $(+a^2)$

It is evident then that the square root of $(+a^2)$ is either $(+a)$ or $(-a)$.

It often happens that the even root of a negative number is expressed.

Ex. $\sqrt{-4}$

Neither $(+a)$ nor $(-a)$ can be used because squaring either $(+a)$ or $(-a)$ will give $(+4)$.

Therefore, we define an indicated even root of a negative number as an *imaginary number.*

Because this kind of a number may appear in an equation at any time, it is necessary to determine a method of simplification.

Given the problem $\sqrt{-16}$

As it stands, the square root cannot be taken,

But by factoring into (16) and (-1), the expression becomes $\sqrt{16 \cdot -1}$

Or $\sqrt{16} \cdot \sqrt{-1}$

This equals $4 \cdot \sqrt{-1}$

It is always possible to take out the factor -1, so that any imaginary number may be thrown into the

form of a radical multiplied by an imaginary number. The radical may then be simplified.

Ex. $3\sqrt{-81}=3\sqrt{81}\cdot\sqrt{-1}=3.9\sqrt{-1}=27\sqrt{-1}$

Ex. $6\sqrt{-49}=6\sqrt{49}\ \sqrt{-1}=6.7\sqrt{-1}=42\sqrt{-1}$

(2) The imaginary number $\sqrt{-1}$ is called the imaginary unit. For convenience the following table should be remembered.

	If $\sqrt{-1}=i$
$(\sqrt{-1})^2=-1$	$i^2=-1$
$(\sqrt{-1})^3=-\sqrt{-1}$	$i^3=-\sqrt{-1}$
$(\sqrt{-1})^4=+1$	$i^4=+1$
$(\sqrt{-1})^5=\sqrt{-1}$	$i^5=\sqrt{-1}$

(*Note*) To shorten the process of working out problems involving imaginary numbers, the letter i is used to stand for $\sqrt{-1}$.

(*Note*) After the factor $\sqrt{-1}$ is taken out, the remaining radical expression is reduced by the laws governing radicals.

Problems

Simplify

1 $\sqrt{-4}$
2 $\sqrt{-16}$
3 $\sqrt{-8}$
4 $\sqrt[4]{-32}$
5 $\sqrt{-28}$
6 $\sqrt{-a^2}$
7 $\sqrt{-x^2y^2}$
8 $\sqrt[4]{-x^4}$
9 $\sqrt{-50}$
10 $\sqrt[3]{-44}$
11 $\sqrt{-4ab^2}$
12 $\sqrt{-m^3n}$
13 $\sqrt[4]{-20}$
14 $\sqrt[6]{-64}$
15 $\sqrt{-18}$

(3) Before applying the fundamental operations to imaginary numbers, it is necessary to write the numbers in equivalent form having the $\sqrt{-1}$ factor expressed.

Ex. Add $\sqrt{-4}$ and $\sqrt{-16}$

$2\sqrt{-1}+4\sqrt{-1}=6\sqrt{-1}$

Ex. Subtract $\sqrt{-a^2}$ from $2\sqrt{-4a^2}$

$(2\cdot 2a\sqrt{-1})-a\sqrt{-1}=4a\sqrt{-1}-a\sqrt{-1}=$
$3a\sqrt{-1}$

Ex. Multiply $\sqrt{-49b^2}$ by $\sqrt{-9a^2b^2}$

$7b\sqrt{-1}\cdot 3ab\sqrt{-1}=21ab^2(\sqrt{-1})^2=$
$21ab^2(-1)=-21ab^2$

Ex. Divide $\sqrt{-25c^4}$ by $\sqrt{-c^2}$

$5c^2\sqrt{-1}\div c\sqrt{-1}=5c$

Often terms may be added or subtracted even though the unit cannot be taken out, providing it is possible to get the even root of some other number.

Ex. $\sqrt{-2}+2\sqrt{-2}-\sqrt{-32}$

$\sqrt{-2}+2\sqrt{-2}-4\sqrt{-2}=-\sqrt{-2}$

(4) An expression involving an integer and an imaginary number is called a complex number.

Ex. $2+\sqrt{-3}$

Problems

Add

1 $\sqrt{-8}+\sqrt{-18}$

2 $\sqrt{-4}+\sqrt{-16}+\sqrt{-9}$

3 $\sqrt{-44}-\sqrt{-99}$

4 $2\sqrt{-16}+3\sqrt{-25}$

5 $\sqrt{-98}-\sqrt{-72}$

6 $3\sqrt{-27}+4\sqrt{-48}$

7 $\sqrt{-12}-\sqrt{-147}$

Simplify

8 $\sqrt{-3}\cdot\sqrt{-1}$

9 $\sqrt{-4}\cdot\sqrt{-3}$

10 $\sqrt{-8}\cdot\sqrt{-3}$

11 $\sqrt{-16}\cdot\sqrt{-4}$

12 $\dfrac{\sqrt{-8}}{\sqrt{-2}}$

13 $\dfrac{\sqrt{-12}}{\sqrt{-27}}$

Lesson 11

Complex numbers

Complex numbers are not difficult to handle, providing one treats all operations involving the imaginary terms with consideration for the preceding rules.

Ex. Add $3+\sqrt{-2}$
$4-\sqrt{-8}$

This may be written
$3+\sqrt{-2}$
$4-2\sqrt{-2}$

Adding $7-\sqrt{-2}$

Ex. Add $6+\sqrt{-9a^2}$
$2+\sqrt{-16a^2}$

Re-writing $6+3a\sqrt{-1}$
$2+4a\sqrt{-1}$

Adding $8+7a\sqrt{-1}$

Ex. Multiply $3-\sqrt{-49}$
$2+\sqrt{-81}$

Re-writing $3-7\sqrt{-1}$
$2+9\sqrt{-1}$

$6-14\sqrt{-1}$
$+27\sqrt{-1}-63(\sqrt{-1})^2$

$6+13\sqrt{-1}-63(\sqrt{-1})^2$

Or, $6+13\sqrt{-1}-63(-1)$
Or, $6+13\sqrt{-1}+63$
Or, $69+13\sqrt{-1}$

Problems

Add

1 $2+\sqrt{-3}$
$3-\sqrt{-3}$

2 $4\sqrt{-8}+2\sqrt{-27}$
$6\sqrt{-18}-2\sqrt{-48}$

3 $6\sqrt{-2}+\sqrt{-32}$
$4\sqrt{-2}-\sqrt{-50}$

Subtract

1 $6-\sqrt{-3}$
$2+\sqrt{-3}$

2 $\sqrt{-3}+1$
$2\sqrt{-3}-4$

3 $6+\sqrt{-72}$
$4-2\sqrt{-2}$

Multiply

1 $5+\sqrt{-3}$
$2-\sqrt{-6}$

2 $3-\sqrt{-6}$
$2+\sqrt{-5}$

3 $5\sqrt{-6}\cdot 2\sqrt{-5}$

4 $6+4\sqrt{-3}$
$2+3\sqrt{-2}$

5 $6+\sqrt{-3}$
$2-\sqrt{-4}$

6 $3-\sqrt{-8}$
$2+\sqrt{-12}$

SECTION V

THE QUADRATIC EQUATION

Lesson 1

Equations in one unknown number.

(1) The equation studied thus far has involved the unknown number or numbers in the first degree only, that is, the exponent of the unknown symbol or symbols has not been greater than one.

(2) The *quadratic equation* is one involving the unknown numbers in the *second degree*, but *no higher degree*. The unknown number *may* appear in the *first degree*.

Ex. $2x^2+3x+6=0$

Ex. $ax^2+bx+c=0$

(3) To classify equations for special solutions it is customary to call an equation involving the unknown number, in the *second degree* only, an *incomplete quadratic equation*, and one involving the unknown number in both the *second and first degrees* a *complete quadratic equation*.

Ex. $3x^2+4=8$ (Incomplete)
$ax^2+c=0$ (Incomplete)

Ex. $4x^2-6x=5$ (Complete)
$ax^2+bx+c=0$ (Complete)

Solutions

The incomplete quadratic equation.

(1) To solve an incomplete quadratic equation it is only necessary to so perform the fundamental operations upon the equation as to get the unknown number upon one side of the equation and the known numbers upon the other side, then extract the square root of both members.

Ex. Given $3x^2+4=8$

Subtracting 4 from both members, $3x^2=4$

Dividing by 3, $x^2=\frac{4}{3}$

Extracting the square root, $x=\pm\sqrt{\frac{4}{3}}$

Or, $x=\pm 2\sqrt{\frac{1}{3}}$

(*Note*) The $\pm$ sign is placed before the number on the right hand side of the equation, owing to the fact that the squaring of either $+\sqrt{\frac{4}{3}}$ or $-\sqrt{\frac{4}{3}}$ will give $\frac{4}{3}$.

Ex. $6x^2+2=26$

Subtracting 2, $6x^2=24$

Dividing by 6, $x^2=4$

Extracting the square root, $x=\pm 2$

(*Note*) The $\pm$ sign is used as before because either $+2$ or -2 squared will give 4.

Ex. $ax^2+b=0$

Subtracting b, $ax^2=-b$

Dividing by a, $x^2=\frac{-b}{a}$

Extracting square root, $x=\pm\sqrt{-\frac{b}{a}}$

(*Note*) The values of the unknown number are called *roots* of the equation.

Problems

Solve,

1 $x^2=16$

2 $3x^2=12$

3 $4x^2+2=18$

4 $\frac{x^2}{2}=4$

5 $\frac{3x^2}{2}=6$

6 $2x^2-1=x^2+3$

7 $2x^2+8=33+x^2$

8 $6x^2-2=22$

9 $8-x^2=-8$

10 $7x^2=343$

11 $x^2+3-2x^2=7$

12 $-x^2+3=-6x^2+48$

Lesson 2

The complete quadratic equation.

(1) The solution of the complete quadratic equation is by the method called completion of the square.

(2) It is based upon the fact that any trinomial having 2 squared terms and the third term equal to twice the square roots of the squared terms is a perfect square.

Ex. $a^2+2ab+b^2$

Ex. x^2+4x+4 etc.

(3) Given an equation,

$x^2+4x+8=0$

It is noticed that one term (x^2) is a square, so by subtracting (8) from both members, $x^2+4x=-8$ we may suppose that a third term might be added to the left hand number, completing a square.

This added term is determined by supposing that $4x$ equals twice the product of the square roots of the other two terms. Then division by two would give the product of the square roots, and again division by one square root (namely x) would give the second square root. Squaring, we have the term to be added.

Example $x^2+4x+\left(\frac{4x}{2x}\right)^2=-8+\left(\frac{4x}{2x}\right)^2$

Simplifying, $x^2+4x+4=-8+4$

Or, $x^2+4x+4=-4$

(*Note*) This last equation has now a perfect square for its left hand member.

Extracting the square root, $x+2=\pm\sqrt{-4}$

Or, $x=\pm\sqrt{-4}-2$

Example $x^2+5x+6=0$

Subtracting 6, $x^2+5x=-6$

Completing square of left hand member,

$$x^2+5x+\left(\frac{5x}{2x}\right)^2=-6+\left(\frac{5x}{2x}\right)^2$$

Simplifying, $x^2+5x+\frac{25}{4}=-6+\frac{25}{4}$

Or, $x^2+5x+\frac{25}{4}=\frac{-24}{4}+\frac{25}{4}$

Or, $x^2+5x+\frac{25}{4}=\frac{1}{4}$

Extracting the square root of both members,

$$x+\frac{5}{2}=\pm\sqrt{\frac{1}{4}}=\pm\frac{1}{2}$$

Subtracting $\frac{5}{2}$, $x=\pm\frac{1}{2}-\frac{5}{2}$

Or, $x=-\frac{4}{2}=-2$ and $-\frac{6}{2}=-3$

(*Note*) The $\pm$ sign signifies that $(\frac{1}{2})$ is to be used first with the $(+)$ sign then with the $(-)$ sign, giving two values for x, $(-2$ or $-3)$.

Example $x^2-x-6=0$

Adding (6) to both members, $x^2-x=+6$

Completing square of left hand member,

$$x^2-x+\left(\frac{x}{2x}\right)^2=+6+\left(\frac{x}{2x}\right)^2$$

Or, $x^2-x+\frac{1}{4}=+6+\frac{1}{4}$

Or, $x^2-x+\frac{1}{4}=\frac{24}{4}+\frac{1}{4}=\frac{25}{4}$

Extracting square root of both members,

$$x-\frac{1}{2}=\pm\frac{5}{2}$$

(*Note*) It will be observed that the term added is always $\left[\frac{\text{2nd term}}{2 \cdot \text{sq. rt. of 1st}}\right]^2$

Adding $(\frac{1}{2})$ to both members, $x=\pm\frac{5}{2}+\frac{1}{2}$

Or, $x=\frac{6}{2}=3$ and $-\frac{4}{2}=-2$

Probl ms

Solve.

1	x^2+5x+6	7	$x^2+8x+15$
2	$x^2+7x+10$	8	$x^2-6x-16$
3	x^2-x-6	9	$x^2+2x-15$
4	$x^2+13x+30$	10	m^2-m-30
5	$a^2+11a+30$	11	$y^2+7y=-12$
6	$b^2-2b-15$	12	$x^2-7x=+18$

13 The sum of the squares of two consecutive numbers is 120. Find the numbers.

14 A lot has an area of one acre. The length is 2 rods more than the width. Find the dimensions.

15 Suppose a field is 80 rods long and 20 rods wide. How wide a strip must be cut to amount to 10 acres.

(4) In case the term containing the square of the unknown number has a coefficient which is not a perfect square, it is first necessary to either multiply or divide by some number to make the coefficient of the squared term a perfect square.

Example $2x^2+3x+4=0$

Dividing by (2), $x^2+\frac{3}{2}x+2=0$

Subtracting (2), $x^2+\frac{3}{2}x=-2$

Completing square, $x^2+\frac{3}{2}x+\left(\frac{3x}{4x}\right)^2=-2+\left(\frac{3x}{4x}\right)^2$

Or, $x^2+\frac{3}{2}x+\frac{9}{16}=-2+\frac{9}{16}=-\frac{23}{16}$

Extracting square root of both members,

$x+\frac{3}{4}=\pm\sqrt{-\frac{23}{16}}$

Or, $x=-\frac{3}{4}\pm\frac{1}{4}\sqrt{-23}$

Example $3x^2+21x+30=0$

Dividing by (3), $x^2+7x+10=0$

Subtracting (10), $x^2+7x=-10$

Completing square, $x^2+7x+\left(\frac{7x}{2x}\right)^2=-10+\left(\frac{7x}{2x}\right)^2=-10+\frac{49}{4}$

Extracting square root, $x+\frac{7}{2}=\pm\sqrt{\frac{9}{4}}=\pm\frac{3}{2}$

Or, $x=\pm\frac{3}{2}-\frac{7}{2}$

Or, $x=-\frac{4}{2}=-2$, and $-\frac{10}{2}=-5$

Problems

1	$2x^2-2x=12$	6	$3y^2-15y-42=0$
2	$3x^2+10x=12$	7	$2x^2+3x+2=0$
3	$2x^2+14x=-20$	8	$3a^2+4a-3=0$
4	$7a^2+7a-42=0$	9	$2m^2+3m-7=0$
5	$2y^2+18y+40=0$	10	$4x^2+3x-2=0$

11 A motor boat goes up stream and back in 6 hours. If the boat goes up 10 miles and the stream has a rate of 4 miles per hour, what is the rate of the motor boat in still water.

12 Sum of squares of two numbers is 73. Their difference is 5. Find the numbers.

(5) The formula.

The equation $ax^2 \pm bx \pm c = 0$ is called the type form of the complete quadratic equation. (a) represents the coefficient of the squared term, (b) represents the coefficient of the first power term, while (c) represents the constant term.

Example—Given the particular equation,

$3x^2 + 2x + 4 = 0$

Then, $a = 3$, $b = 2$, $c = 4$

If, then, we solve the equation,

$ax^2 + bx + c = 0$,

we solve the type of all quadratic equations. It is then possible to substitute the particular numbers for (a), (b), and (c) in the result and get a solution for any particular equation.

Example—Given the equation

$ax^2 + bx + c = 0$

Solving this by completing the square, we get

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(*Note*) This solution is given later.

Now suppose the particular equation is

$x^2 + 8x + 15 = 0$

$a = 1$, $b = 8$, $c = 15$

Substituting these values of a, b, and c in the solution of the type we get

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-8\pm\sqrt{64-(4.1.15)}}{2}=$$

$$\frac{-8\pm\sqrt{64-60}}{2}=\frac{-8\pm\sqrt{4}}{2}=\frac{-8\pm2}{2}=$$

$$\frac{-10}{2}\text{ or }\frac{-6}{2}\text{ or }-5\text{ or }-3$$

Example—Given the problem $3x^2-5x+6=0$

We know that the solution of the type gives

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

In the given problem,

$a=3,\ b=-5,\ c=6$

Substituting, $x=\dfrac{-(-5)\pm\sqrt{(-5)^2-(4.3.6)}}{2.3}$

Simplifying, $x=\dfrac{+5\pm\sqrt{25-72}}{6}$

Or, $x=\dfrac{+5\pm\sqrt{-47}}{6}$

Example—Given $x^2+6x+8=0$

$a=1,\ b=6,\ c=8$

Type form, $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

Substituting, $x=\dfrac{-6\pm\sqrt{(6)^2-(4.1.8)}}{2}$

Or, $x=\dfrac{-6\pm\sqrt{4}}{2}=\dfrac{-6\pm2}{2}$

$x=\dfrac{-8}{2}=-4$ or $\dfrac{-4}{2}=-2$

(*Note*) The given equation must be arranged so that all terms are upon the left side of the equality sign, that is, arranged like the type form.

The solution of the type form follows.

Problems

Solve by using the formula.

1 $x^2+10x+21=0$

2 $2x^2+24x+54=0$

3 $3x^2-3x-90=0$

4 $5a^2-3a+2=0$

5 $2y^2+3y-4=0$

6 $5b^2+3b+2=0$

7 $3m^2+4m+3=0$

8 $6y^2+2y=-6$

9 $5m^2+2m-5=0$

10 $3a^2-2+3a=0$

11 $3b^2-2b=6$

12 $4c^2=-5c+2$

Lesson 5

Development of the formula

(6) Given, $ax^2+bx+c=0$

Subtracting (c), $ax^2+bx=-c$

Multiplying both members by (a), $a^2x^2+abx=-ac$

Completing square of left hand member,

$$a^2x^2+abx+\left(\frac{abx}{2ax}\right)^2=-ac+\left(\frac{abx}{2ax}\right)^2$$

Or, $$a^2x^2+abx+\frac{b^2}{4}=-ac+\frac{b^2}{4}$$

Extracting square root of both members,

$$ax+\frac{b}{2}=\pm\sqrt{-ac+\frac{b^2}{4}}$$

Or, $$ax+\frac{b}{2}=\pm\sqrt{\frac{-4ac+b^2}{4}}$$

Subtracting $\frac{b}{2}$, $$ax=-\frac{b}{2}\pm\sqrt{\frac{-4ac+b^2}{4}}$$

Simplifying, $$ax=-\frac{b}{2}\pm\frac{1}{2}\sqrt{-4ac+b^2}$$

Or, $$ax=\frac{-b\pm\sqrt{-4ac+b^2}}{2}$$

Dividing by (a), $$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

(*Note*) This solution may be obtained by division of both members by the coefficient of (x^2), or in fact by multiplying or dividing by any number making the (x^2) term a perfect square.

(7) If the solution of the type form of the quadratic equation gives,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

it is evident that there is in every solution of particular problems a value corresponding to $b^2 - 4ac$, and it is also evident that it is this value that determines to a great extent the value of the quadratic equation.

When substitutions are made, if $b^2 - 4ac$ is *positive*, then the quantity under the radical sign is *positive*, and the values of (x) are

real

unequal (due to $\pm$ sign).

When substitutions are made, if $b^2 - 4ac$ is *negative*, then the quantity under the radical sign is *negative*, and the values of (x) are

imaginary

unequal (due to $\pm$ sign).

When substitutions are made, if $b^2 - 4ac$ is *zero*, then the quantity under the radical sign is *zero*, causing the radical term to drop out, making the values of x

real

equal (The term preceded by the $\pm$ sign drops out.)

(8) This value, $b^2 - 4ac$, is called the *discriminant*, and by considering its value it is possible to determine a great deal about the values of the equation before the solution is completed.

Example $3x^2+5x+6=0$

$a=3,\ b=5,\ c=6,$

$b^2-4ac=25-72=-47$

-47 has the $(-)$ sign preceding it so we know the values of (x) are

imaginary

unequal

Or, $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}=\dfrac{-3\pm\sqrt{-47}}{6}$ imaginary unequal

Example $2x^2+4x-7=0$

$a=2,\ b=4,\ c=-7$

$b^2-4ac=16-(4.2.-7)$

$=16-(-56)$

$=16+56$

$=72$

$+72$ has the $(+)$ sign preceding it, so we know the values of (x) are

real

unequal

Or $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}=\dfrac{-4\pm\sqrt{72}}{4}$ real unequal

Example $x^2+4x+4=0$

$a=1,\ b=4,\ c=4$

$b^2-4ac=16-(4.1.4)$

$b^2-4ac=16-16=0$

$b^2-4ac=0$, so we know that the values of (x) are

real

equal

Or, $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}=\dfrac{-4\pm0}{2}=-2$ or -2

Problems

Discuss the following problems with reference to the character of the roots.

1	$3x^2+2x-3=0$	6	$y^2+3y-6=0$
2	$6x^2-3x+4=0$	7	$2y^2+3y-2=0$
3	$x^2-2x+6=0$	8	$3m^2+2m-6=0$
4	$a^2-3a+4=0$	9	$6y^2+5y-7=0$
5	$2a^2+2a-6=0$	10	$2r^2+3r+8=0$

Lesson 6

Special Solution by Factoring

(9) In case the problem can be arranged so that the left hand member is factorable, the solution is easily accomplished.

Example $x^2-x-6=0$

Factoring left hand member, $(x-3)\ (x+2)=0$

Because the product of the two factors equals zero, it is evident that at least one factor must equal zero. Inasmuch as either factor equaling zero would cause the product to become equal to zero,

$(x-3)$ may equal zero.

Or $(x+2)$ may equal zero.

If $x-3=0$

$x=3$

If $x+2=0$

$x=-2$

Either of these values for (x) will satisfy the equation.

Example $x^2+5x+6=0$

Factoring, $(x+3)\ (x+2)=0$

If $x+3=0$, $x=-3$

If $x+2=0$, $x=-2$

Example $x^2+7x+10=0$

Factoring, $(x+2)\ (x+5)=0$

If $x+2=0$, $x=-2$

If $x+5=0$, $x=-5$

Example $x^3+6x^2+12x+8=0$

Factoring, $(x+2)^3=0$

If $x+2=0$, $x=-2$

The other two roots are also equal to -2.

(*Note*) This last equation is not a quadratic equation, but may be solved by factoring.

Problems

1	$2x+1=0$	7	$2a^2+34a+144=0$
2	$x^2+10x+21=0$	8	$2(x^2-4)=0$
3	$a^2+9x+18=0$	9	$3x^2+21x+30=0$
4	$m^2-5m-6=0$	10	$x^2+10x+25=0$
5	$x^2+14x+24=0$	11	$a^2+14a+48=0$
6	$5x^2+25x+30=0$	12	$x^2-1=0$

Lesson 7

(10) A simple test for the solution of a quadratic equation is in the fact that the sum and the product of the roots have a definite relation to the coefficients and the constant.

The relationship is,

Given the equation, $ax^2+bx+c=0$

Sum of roots $=-\frac{b}{a}$

Product of roots $=\frac{c}{a}$

Example—Suppose the equation, $x^2+8x+15=0$ is solved giving the values, $x=-3$

$x=-5$

Sum of roots $=-8=-\frac{8}{1}$

Product of roots $=+15=\frac{15}{1}$

Problems

Solve and test results.

1	$x^2+17x+30=0$	6	$6-3x^2+2x=0$
2	$x^2+27x+50=0$	7	$4-3x=-4x^2$
3	$a^2-10a-39=0$	8	$16x^2-4x-2=0$
4	$y^2+10y+24=0$	9	$r^2+11r+10=0$
5	$m^2+13m+22=0$	10	$2-4x^2+3x=0$

(11) The proof of this relationship is demonstrated as follows:

Given the equation,

$$ax^2+bx+c=0$$

Solving by formula,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The values of x are,

(1) $$x=\frac{-b+\sqrt{b^2-4ac}}{2a}$$

$$x^1=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

Adding, $$(x+x^1)=\frac{-2b}{2a}=\frac{-b}{a}$$

(2) $$x=\frac{-b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}$$

$$x^1=\frac{-b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}$$

Multiplying, $$xx^1=\frac{b^2}{4a^2}-\frac{b^2-4ac}{4a^2}$$

(*Note*) It will be noticed that the two roots represent the sum and difference of the two quantities

$$\frac{-b}{2a} \text{ and } \frac{b^2-4ac}{2a}$$

The product then is the difference of the squares.

Combining, $$xx^1=\frac{b^2-b^2+4ac}{4a^2}=\frac{4ac}{4a^2}=\frac{c}{a}$$

Problems

In the following problems determine the sum and product of the roots.

1 $2x^2+3x-4=0$

2 $x^2+4x+4=0$

3 $3x^2+2x-4=0$

4 $5a^2+6a-2=0$

5 $3m^2-3m+2=0$

6 $6a^2+2a-3=0$

7 $3b^2+2b-6=0$

8 $5m^2+6=3m$

9 $2m^2=16-4m$

10 $\frac{m^2}{2}-3m+6=0$

Lesson 9

Higher degree equations

(12) Certain higher degree equations may be solved by quadratic methods, provided the equations can be placed in the quadratic form.

Example $x^4+12x^2+20=0$

Solving for (x^2) instead of (x),

$$x^2=\frac{-12\pm\sqrt{144-80}}{2}$$

$$x^2=\frac{-12\pm\sqrt{64}}{2}=\frac{-12\pm 8}{2}=$$

$$\frac{-20}{2}=-10 \text{ or } \frac{-4}{2}=-2$$

$$x=\pm\sqrt{-10} \text{ or } \pm\sqrt{-2}$$

(*Note*) In such a problem the unknown appears in (x^2) and (x^4), the one the square of the other.

Example $2x^6-2x^3-12=0$

Dividing by 2, $x^6-x^3-6=0$

$$x^3=\frac{-1\pm\sqrt{1+24}}{2}$$

$$x^3=\frac{-1\pm 5}{2}=\frac{-6}{2} \text{ or } \frac{+4}{2}$$

$x^3=-3$ or 2

Therefore $x=\sqrt[3]{-3}$ or $\sqrt[3]{2}$

Problems

Solve.

1	$x^4+5x^2+6=0$	6	$y^6+12y^3+11=0$
2	$2x^4-2x^2-20=0$	7	$5x^4+2x^2-2=0$
3	$y^6+y^3-56=0$	8	$3m^3-2m^4=10$
4	$6x^4=24$	9	$6r^4+3r^2=2$
5	$a^4+15a^2+26=0$	10	$2m^4=32$

Indeterminate Quadratic Equations

1 An indeterminate equation is satisfied by an infinite number of sets of values as was shown in the previous discussion of simple indeterminate equations. The solutions discussed had to do with the simultaneous solutions of two or more equations, that is, the determination of a set of values that would satisfy both or all the equations under discussion. It was found that given simple equations could be solved simultaneously by use of one of three methods, namely, addition or subtraction, substitution, or comparison. These same methods may be used when second degree equations are involved. When one or both of two equations are of the second degree, it is not always possible to get a solution by the use of one of the three methods mentioned, so special methods are employed for special groups of equations. There special methods will be discussed under special cases in the advanced course.

Graphic Representations

1 The graphic representation of an indeterminate quadratic equation is accomplished by a method similar to that used for the simple indeterminate equation. The fact that the unknown has two values makes possible the location of two points in each set of values.

Ex. Plot $2x^2+y^2=4$

Determining the sets of values,

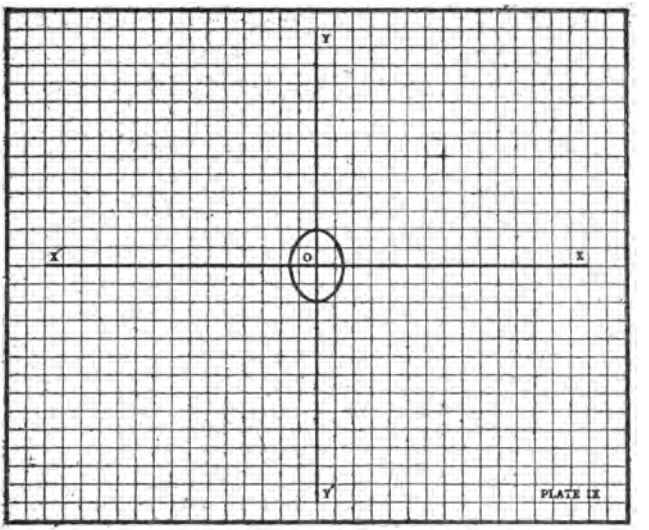
If $x=0$, $y=\pm 2$

If $x=1$, $y=\pm\sqrt{2}$

If $x=2$, $y=\pm\sqrt{-4}$

If $x=-1$, $y=\pm\sqrt{2}$
If $x=-2$, $y=\pm\sqrt{-4}$
If $x=\frac{1}{2}$, $y=\pm\sqrt{\frac{7}{2}}$
If $x=\frac{1}{3}$, $y=\pm\sqrt{\frac{34}{9}}=+\frac{1}{3}\sqrt{34}$

Plotting these sets of values,



(*Note*) Imaginary values cannot be plotted.

In the graph it is noticed that when $x=1$ there are two values of y to locate, one above the x-axis, and one below the x-axis. So with all other values.

Ex. Plot, $y^2=5x+2$

Determining sets of values,

If $x=0$, $y=\pm\sqrt{2}$
If $x=1$, $y=\pm\sqrt{7}$
If $x=2$, $y=\pm\sqrt{12}$